Keisler-Shelah theorem and (higher) cardinal invariants

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Keisler–Shelah theorem

Keisler–Shelah theorem

For every (first-order) language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$, the following are equivalent:

1. $\mathcal{A} \equiv \mathcal{B}$ (that is, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).
2. There is a nonprincipal ultrafilter $\mathcal{U}$ over an infinite set such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

(2) $\Rightarrow$ (1) is obvious. Keisler proved (1) $\Rightarrow$ (2) under GCH. Shelah eliminated GCH assumption.

How about versions with restrictions on the cardinalities of languages, structures and the underlying sets of ultrafilters?

Keisler–Golshani–Shelah theorem

Keisler–Golshani–Shelah theorem (Keisler, Golshani and Shelah)

The following are equivalent:

1. The continuum hypothesis.
2. For every countable language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leqslant \mathfrak{c}$, if $\mathcal{A} \equiv \mathcal{B}$ then there is a nonprincipal ultrafilter $\mathcal{U}$ over ω such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

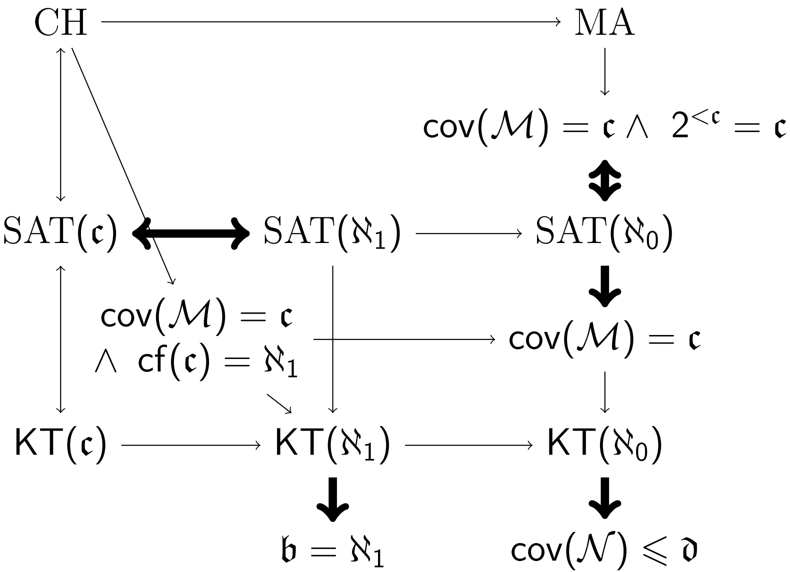
(1) $\Rightarrow$ (2) was proved by Keisler (1961) and (2) $\Rightarrow$ (1) is due to Golshani and Shelah (2023).

The principles

Let λ be a cardinal.

1. We say KT(λ) holds if for every countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leqslant \lambda$ which are elementarily equivalent, there exists an ultrafilter $\mathcal{U}$ over ω such that $\mathcal{A}^\omega/\mathcal{U} \simeq \mathcal{B}^\omega/\mathcal{U}$.
2. We say $\mathrm{SAT}(\lambda)$ holds if there exists an ultrafilter $\mathcal{U}$ over ω such that for every countable language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leqslant \lambda$, $\prod_{i\in\omega} \mathcal{A}_i/\mathcal{U}$ is saturated.

The implications (thick lines are due to the speaker)



Let's generalize these principles much more!

The generalized principles

Let κ, μ and λ be infinite cardinals.

$\mathsf{KT}(\kappa; \mu, \lambda) \iff$ for every language $\mathcal{L}$ of size $\leqslant \mu$ and every elementarily equivalent $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leqslant \lambda$, there is a uniform ultrafilter $\mathcal{U}$ on κ s.t. $\mathcal{A}^{\mathcal{U}} \simeq \mathcal{B}^{\mathcal{U}}$.

$\mathsf{SAT}(\kappa; \mu, \lambda) \iff$ there is a uniform ultrafilter $\mathcal{U}$ on κ such that for every language $\mathcal{L}$ of size $\leqslant \mu$ and every sequence $\langle \mathcal{A}_i : i < \kappa \rangle$ of infinite $\mathcal{L}$-str. of size $\leqslant \lambda$, the ultraproduct $\left(\prod_{i \in \kappa} \mathcal{A}_i \right) / \mathcal{U}$ is saturated.

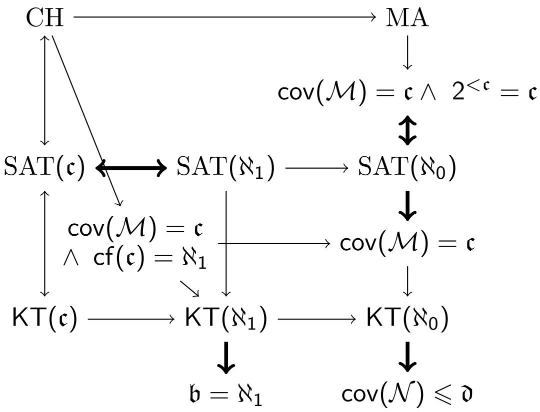
Keisler-Shelah theorem in this context

The statement of Keisler-Shelah theorem can be said as $\mathrm{KT}(2^\kappa; 2^\kappa, \kappa)$.

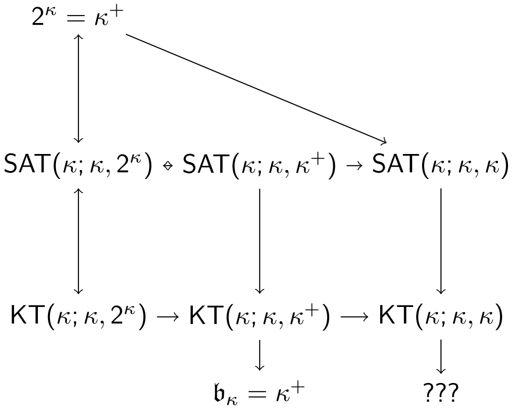
$\mathrm{KT}(\text{index set of ultrafilter}; \text{size of language}, \text{size of structures})$

The implications

The countable case



The general case



Results

KT(index set of ultrafilter; size of language, size of structures)

1. SAT$(\kappa;\mu,\lambda)$ implies KT$(\kappa;\mu,\lambda)$.
2. $\neg$ SAT$(\kappa;\aleph_0,\kappa^{++})$.
3. $\neg$ KT$(\kappa;\aleph_0,\kappa^{++})$.
4. SAT$(\kappa;\aleph_0,\kappa^{+})$ implies $2^\kappa=\kappa^+$.
5. The following are equivalent.

   a. $2^\kappa=\kappa^+$. b. SAT$(\kappa;\mu,2^\kappa)$. c. SAT$(\kappa;\mu,\kappa^+)$. d. KT$(\kappa;\mu,2^\kappa)$.

6. SAT$(\kappa;\aleph_0,\kappa)$ implies $2^{<2^\kappa}=2^\kappa$.
7. When κ is regular, KT$(\kappa;\aleph_0,\kappa^{+})$ implies $\mathfrak{b}_\kappa=\kappa^+$.

The meager ideal on κ

Let κ be a regular cardinal. Topologize $2^\kappa = \{0,1\}^\kappa$ by $<\kappa$-box topology, where $\{0,1\}$ is the discrete space. Then the meager ideal on 2^κ is κ-additive ideal generated by nowhere dense sets of 2^κ.

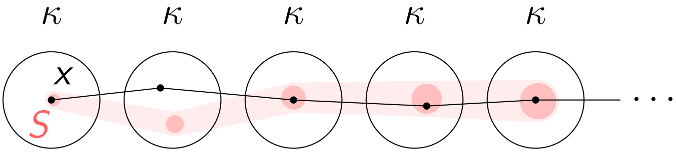
Let $\mathrm{cov}(\mathcal{M}_\kappa)$ be the covering number of the meager ideal of 2^κ.

Results related to $\mathrm{cov}(\mathcal{M}_\kappa)$

1. When κ is a regular cardinal, $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$ implies $\mathrm{KT}(\kappa; \mu, \kappa)$ for $\mu < 2^\kappa$.
2. When κ is an **inaccessible** cardinal, $\mathrm{SAT}(\kappa; \aleph_0, \kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.
   - We showed this result by using van der Vlugt's theorem extending Bartoszyński's characterization of $\mathrm{cov}(\mathcal{M})$ in terms of slaloms.
3. When κ is a regular cardinal, $\mathrm{cov}(\mathcal{M}_\kappa) = 2^{<2^\kappa} = 2^\kappa$ implies $\mathrm{SAT}(\kappa; \kappa, \kappa)$.

Results related to $\mathrm{cov}(\mathcal{M}_\kappa)$

Bartoszyński–van der Vlugt theorem Let κ be an inaccessible cardinal. Then $\mathrm{cov}(\mathcal{M}_\kappa) \geqslant \lambda$ holds iff for every $X \subseteq \kappa^\kappa$ of size $<\lambda$ there is $S \in \prod_{i<\kappa}[\kappa]^{\leqslant(|i|+1)}$ for all $x \in X$ we have $\{i < \kappa : x(i) \in S(i)\}$ is cofinal in κ.



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Results related to $\mathrm{cov}(\mathcal{M}_\kappa)$

Theomem (G.) Let κ be inaccessible. Then $\mathrm{SAT}(\kappa; \aleph_0, \kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

Let $\mathcal{U}$ be a regular ultrafilter on κ witnessing $\mathrm{SAT}(\kappa; \aleph_0, \kappa)$. Let $X \subseteq \kappa^\kappa$ of size $<2^\kappa$. Let $\mathcal{L} = \{\subseteq\}$. For $i < \kappa$, define a $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\kappa]^{<|i|}, \subseteq)$. For $x \in \kappa^\kappa$, we define $S_x = \langle \{x(i)\} : i < \kappa \rangle$. Put $\mathcal{A}_* = \prod_{i<\kappa} \mathcal{A}_i / \mathcal{U}$. Consider a set of formulas $p(S)$ defined by

$$p(S) = \{\ulcorner [S_x] \subseteq S \urcorner : x \in X\}.$$

Then $p(S)$ is finitely satisfiable and number of parameters occurring in $p(S)$ is $<2^\kappa$. Thus, by $\mathrm{SAT}(\kappa; \aleph_0, \kappa)$, we can take $[S] \in \mathcal{A}_*$ realizing $p(S)$.

Results related to $\mathrm{cov}(\mathcal{M}_\kappa)$

Then we have

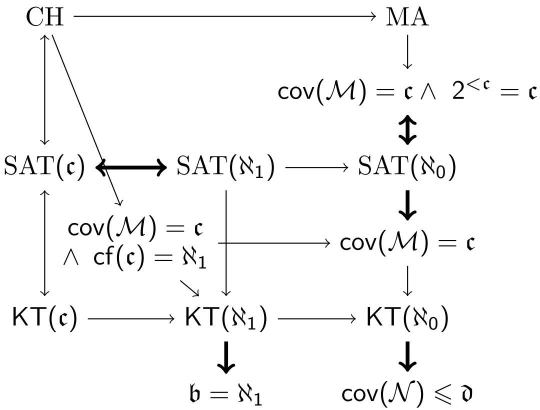
$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \in \mathcal{U}).$$

But since our ultrafilter $\mathcal{U}$ is uniform, we have

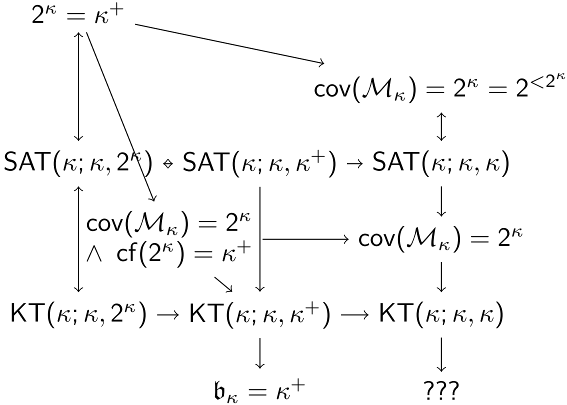
$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \text{ is cofinal}).$$

So by van der Vlugt's theorem, we showed $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

The countable case

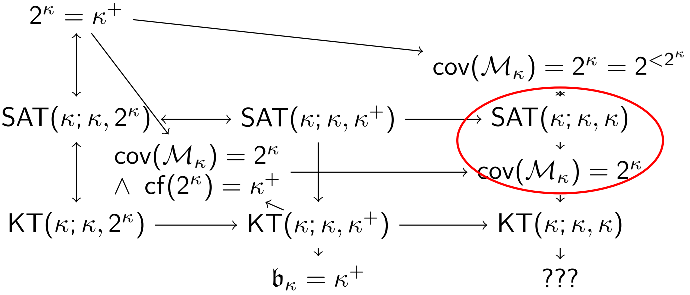


The **inaccessible** case



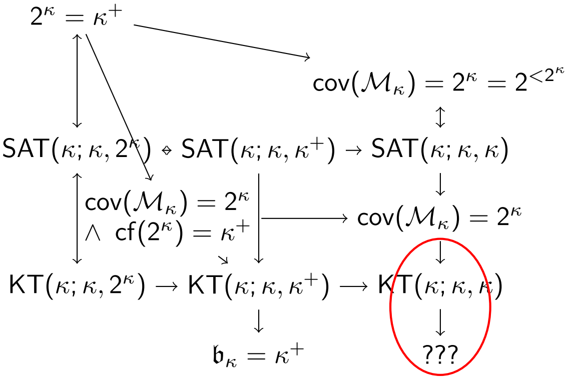
Questions

1. Can we eliminate the inaccessibility assumption from the result which states $\mathrm{SAT}(\kappa; \aleph_0, \kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.
    - Note that when κ is a successor and $2^{\kappa^-} > \kappa$, we have $\mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$.
    - Note also that when κ is a successor, "The minimum cardinality of $X \subseteq \kappa^\kappa$ such that there is no $S \in \prod_{i<\kappa} [\kappa]^{\leqslant |i|+1}$ such that for all $x \in X$, $\{i < \kappa : x(i) \in S(i)\}$ is cofinal in κ" is equal to $\mathfrak{d}_\kappa$.



Questions

2. Can we prove the consistency of $\neg\,\mathrm{KT}^{\kappa}_{\kappa}(\kappa)$?
   - Recall that $\mathrm{KT}^{\aleph_0}_{\aleph_0}(\aleph_0)$ implies $\mathrm{cov}(\mathcal{N}) \leqslant \mathfrak{d}$.



Questions for the countable structures

1. Does $\mathrm{KT}(\aleph_1)$ imply $\mathrm{non}(\mathcal{M}) = \aleph_1$?
2. Does $\mathrm{KT}(\aleph_0)$ imply $\mathrm{non}(\mathcal{M}) \leqslant \mathrm{cov}(\mathcal{M})$?
3. In the Sacks model, does $\mathrm{KT}(\aleph_0)$ hold?

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