Approaches to open problems regarding Goldstern's principle

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partially joint work with Martin Goldstern

Goldstern's theorem

In 1993, Martin Goldstern proved the following theorem.

Goldstern's theorem

Let $A \subseteq \omega^\omega \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$, A_x has Lebesgue measure 0. Also, assume $(\forall x, x' \in \omega^\omega)(x \leqslant x' \Rightarrow A_x \subseteq A_{x'})$, which is called monotonicity property. Then $\bigcup_{x \in \omega^\omega} A_x$ has also Lebesgue measure 0.

Here, $\leqslant$ on ω^ω is total domination order and A_x is the vertical section of A at x.

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Here, $\leqslant$ on ω^ω is total domination order and A_x is the vertical section of A at x.

He used Shoenfield absoluteness theorem and the random forcing to show this theorem. Also he applied this theorem to uniform distribution theory.

The principle GP(Γ)

Definition

Let Γ be a pointclass. Then GP(Γ) means the following statement: Let $A \subseteq \omega^\omega \times 2^\omega$ be in Γ. Assume that for each $x \in \omega^\omega$, A_x has Lebesgue measure 0. Also suppose the monotonicity property. Then $\bigcup_{x\in\omega^\omega} A_x$ has also Lebesgue measure 0.

The principle $\mathrm{GP}(\Gamma)$

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Goldstern's theorem says that $\mathrm{GP}(\mathbf{\Sigma}^1_1)$ holds.

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Goldstern's theorem says that $\mathrm{GP}(\boldsymbol{\Sigma}^1_1)$ holds.
Note that if we replace the domination order $\leqslant$ by the almost domination order $\leqslant^*$, then the principle does not change.

Theorems the speaker showed in the previous paper

The symbol “all” denotes the class of all subsets of Polish spaces.

Theorem (G.)

GP(all) is independent from ZFC.

In particular, Laver model satisfies GP(all).
If $\mathfrak{d} < \mathrm{add}(\mathcal{N})$ were consistent, then the consistency of GP(all) would be easy.
But since $\mathrm{add}(\mathcal{N}) \leqslant \mathfrak{d}$ is provable, this theorem is not trivial.

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Theorem (G.)

GP($\mathbf{\Pi}^1_1$) holds.

1

GP with large continuum

2

GP in Mathias model

(joint with M. Goldstern)

3

GP and Borel conjecture

4

Hausdorff measure version of GP

1

GP with large continuum

Conjecture

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Conjecture

If GP(all) holds then $\mathbb{B}_\kappa \Vdash$ GP(all) for all κ, where $\mathbb{B}_\kappa$ is the random forcing adding κ-many random reals.

Partial result

Partial result

If $\mathfrak{b} = \mathfrak{d}$ and GP(all) hold then $\mathbb{B} \Vdash$ GP(all), where $\mathbb{B}$ is the random forcing adding one random real.

Partial result: proof

Partial result If $\mathfrak{b} = \mathfrak{d}$ and GP(all) hold then $\mathbb{B} \Vdash$ GP(all).

The proof of this was suggested by Jörg Brendle.
Note that if $\mathfrak{b} = \mathfrak{d} = \kappa$, then GP(all) is equivalent to the following statement: for every increasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of null sets, the union of this sequence is also null.
Let r be a random real over V and work in $V[r]$. Fix an increasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of null sets of length $\mathfrak{b} = \mathfrak{d} = \kappa$. For each α, take a Borel null set $B_\alpha \subseteq 2^\omega \times 2^\omega$ coded in V such that $A_\alpha \subseteq (B_\alpha)_r$.
By GP(all) in V, we have $\bigcup_{\alpha<\kappa} \bigcap_{\beta\in[\alpha,\kappa)} B_\beta$ is null. On the other hand, we have $\bigcup_{\alpha<\kappa} A_\alpha \subseteq \bigcup_{\alpha<\kappa} \bigcap_{\beta\in[\alpha,\kappa)} (B_\beta)_r$. Therefore, $\bigcup_{\alpha<\kappa} A_\alpha$ is also null. □

Attempt to the conjecture

Conjecture If GP(all) holds then $\mathbb{B}_{\omega_3} \Vdash \text{GP(all)}$.

Lemma 1: Weak Delta System Lemma (folklore)

Assume $\mathrm{cf}([\alpha]^{<\lambda}, \subseteq) < \kappa$ for every $\alpha < \kappa$. Then for every $\langle X_\alpha : \alpha < \kappa \rangle$ if each X_α has size $<\lambda$, then there is a R of size $<\lambda$ and $\Omega \in [\kappa]^\kappa$, we have $X_\alpha \cap X_\beta \subseteq R$ for every distinct $\alpha, \beta \in \Omega$.

Lemma 2 (G. with Google Gemini)

Let κ be an uncountable cardinal and G be a $(V, \mathbb{B}_\kappa)$-generic filter. Work in $V[G]$. Let r_i $(i < \kappa)$ be the i-th random real. Consider a sequence $\langle A_i : i < \kappa \rangle$ of Borel null sets such that A_i is coded in $V[r_i]$. Then $\bigcap_{i<\kappa} A_i$ is covered by a Borel null set coded in V.

Attempt to the conjecture

Let G be a $(V, \mathbb{B}_{\omega_3})$-generic filter and work in $V[G]$. Let $\langle r_i : i < \kappa \rangle$ be the sequence of random reals. Fix an increasing sequence $\langle A_\alpha : \alpha < \mathfrak{b} \rangle$ of null sets of length $\mathfrak{b} = \mathfrak{d}$.
For each α, take a countable set $I_\alpha \subseteq \kappa$ in V and a Borel null set $B_\alpha \subseteq (2^\omega)^{I_\alpha} \times 2^\omega$ coded in V such that $A_\alpha \subseteq (B_\alpha)_{\langle r_i : i \in I_\alpha \rangle}$.
Applying the weak delta system lemma, we can assume that there is a countable set $J \subseteq \kappa$ in V such that $I_\alpha \cap I_\beta \subseteq J$ for every distinct $\alpha, \beta < \kappa$.
We consider the intermediate model $V' := V[r_i : i \in J]$.
For each $\alpha < \mathfrak{b}$, we can take a Borel null set C_α coded in V', $\bigcap_{\beta \in [\alpha, \mathfrak{b})} (B_\beta)_{\langle r_i : i \in I_\beta \rangle} \subseteq C_\alpha$ by Lemma 2.

Attempt to the conjecture

By GP(all) in V', we have $\bigcup_{\alpha<\mathfrak{b}} \bigcap_{\beta\in[\alpha,\mathfrak{b})} C_\beta \in \mathcal{N}$.
But we have

$$\begin{aligned}\mathcal{N} \ni & \bigcup_{\alpha<\mathfrak{b}} \bigcap_{\beta\in[\alpha,\mathfrak{b})} C_\beta \\ \supseteq & \bigcup_{\alpha<\mathfrak{b}} \bigcap_{\beta\in[\alpha,\mathfrak{b})} \bigcap_{\gamma\in[\beta,\mathfrak{b})} (B_\gamma)_{\langle r_i : i\in I_\gamma\rangle} \\ = & \bigcup_{\alpha<\mathfrak{b}} \bigcap_{\beta\in[\alpha,\mathfrak{b})} (B_\beta)_{\langle r_i : i\in I_\beta\rangle} \\ \supseteq & \bigcup_{\alpha<\mathfrak{b}} A_\alpha\end{aligned}$$

This finishes the proof...?

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This holds in V', but not in $V[G]$!

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\end{aligned}$$

This finishes the proof...? NO!

GP in Mathias model

(joint with M. Goldstern)

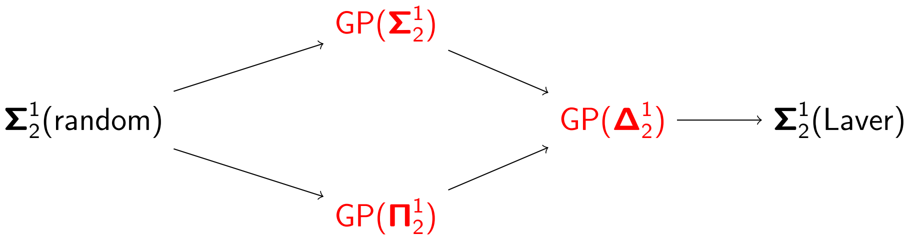
2

GP(all) in Mathias model

GP(all) implies $\mathfrak{b} \neq \mathrm{non}(\mathcal{N})$ and Mathias model (the model obtained by countable support iteration of Mathias forcing of length ω_2) satisfies $\mathfrak{b} = \mathrm{non}(\mathcal{N})$. Therefore, in Mathias model, GP(all) fails.

Then, to what extent does the GP at the projective level hold in this model?

ZFC results


$\mathbf{\Sigma}^1_2$(random)
GP($\mathbf{\Sigma}^1_2$)
GP($\mathbf{\Pi}^1_2$)
GP($\mathbf{\Delta}^1_2$)
$\mathbf{\Sigma}^1_2$(Laver)

Models

$\mathbb{B} :=$ random, $\mathbb{L} :=$ Laver.

	$\mathbf{\Sigma}^1_2(\mathbb{B})$	$\mathbf{\Delta}^1_2(\mathbb{B})$	$\mathbf{\Sigma}^1_2(\mathbb{L})$	GP(all)	GP($\mathbf{\Sigma}^1_2$)	GP($\mathbf{\Pi}^1_2$)	GP($\mathbf{\Delta}^1_2$)
Cohen	NO	NO	NO	NO	NO	NO	NO
random	NO	YES	NO	NO	NO	NO	NO
amoeba	YES	YES	YES	NO	YES	YES	YES
Laver	NO	NO	YES	YES	YES	YES	YES
Hechler	NO	NO	YES	NO	?	?	?
Mathias	NO	NO	YES	NO	?	?	?
Laver*random	NO	YES	YES	YES	YES	YES	YES
Hechler*random	NO	YES	YES	NO	?	?	?
Mathias*random	NO	YES	YES	NO	?	?	?

Conjecture

Conjecture

$\mathrm{GP}(\boldsymbol{\Pi}^1_2)$ fails in Mathias model over L.

In Mathias model over L, consider the set A defined by:

$$A_x = \{y \in 2^\omega : x \text{ dominates } L[y]\}.$$

It is easy to show A is $\boldsymbol{\Pi}^1_2$, $\langle A_x : x \in \omega^\omega \rangle$ is monotone and $\bigcup_{x \in \omega^\omega} A_x = 2^\omega$. But it is not clear that each A_x is null.

Conjecture

Sub Conjecture

A_{m_α} is equal to $L[G_\alpha] \cap 2^\omega$, where m_α is α-th Mathias real.

($A_x = \{y \in 2^\omega : x \text{ dominates } L[y]\}$.)

Conjecture

Sub Conjecture

A_{m_α} is equal to $L[G_\alpha] \cap 2^\omega$, where m_α is α-th Mathias real.

($A_x = \{y \in 2^\omega : x$ dominates $L[y]\}$.)

Partial Result (Goldstern–G.)

(One-step) Mathias forcing forces that $\forall x \in 2^\omega \setminus V \; \exists y \in \omega^\omega \cap V[x] \; \dot{m} < y$.

We want to extend this result to iteration of Mathias forcing.

Proof outline of the partial result

Partial Result (Goldstern–G.)

$\mathbb{M} \Vdash \forall x \in 2^\omega \setminus V \ \exists y \in \omega^\omega \cap V[x] \ \dot{m} < y.$

(sketch) Use pure decision repeatedly to determine the initial segment of x in each finite initial segment of Mathias real. Using information such as the smallest index where the initial segment diverges from the actual x, one can construct y from x that dominates the Mathias real. □

3 GP and Borel conjecture

Borel conjecture

A set $A \subseteq \mathbb{R}$ is called **strong measure zero** if for every sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive real numbers, there is a sequence $\langle I_n : n \in \omega \rangle$ of intervals such that length of I_n is less than ε for every n and $A \subseteq \bigcup_n I_n$.

Borel conjecture states that every strong measure zero set is countable.

Laver model and Mathias model satisfy Borel conjecture.

As we saw, in Mathias model, GP(all) fails. Thus Borel conjecture does not imply GP(all).

How is the converse: does GP(all) imply Borel conjecture?

Conjecture about Borel conjecture

Conjecture

GP(all) does not imply the Borel conjecture.

A candidate model is one obtained by countable support iteration of Zapletal's club shooting poset. We have showed GP(all) holds in the model. But we don't know whether the Borel conjecture holds in it.

The negation of the conjecture

Negation of Conjecture

GP(all) implies the Borel conjecture.

Partial result (G.)

GP(all) implies $\mathcal{SN} = \mathcal{NA}$.

Here, $\mathcal{SN}$ is the strong measure zero ideal and $\mathcal{NA}$ is the null additive ideal. Note that:

$$[\mathbb{R}]^{\leqslant \aleph_0} \subseteq \mathcal{NA} \subseteq \mathcal{SN}.$$

Partial result: proof

Recall that $\mathcal{NA} := \{X \subseteq \mathbb{R} : \forall N \in \mathcal{N} \; X + N \in \mathcal{N}\}$.
Also, there is the characterization of $\mathcal{SN}$:

$$\mathcal{SN} = \{X \subseteq \mathbb{R} : \forall E \in \mathcal{E} \; X + E \in \mathcal{N}\}.$$

Here, $\mathcal{E}$ is the σ-ideal generated by closed measure zero sets.

Partial result: proof

IP denotes the set of interval partition.

Lemma

For every $N \in \mathcal{N}$, there is a monotone family $\langle E_J : J \in \mathrm{IP} \rangle$ of members in $\mathcal{E}$ such that $N \subseteq \bigcup_{J \in \mathrm{IP}} E_J$.

Proof. Take a sequence $\langle C_n : n \in \omega \rangle$ of clopen sets such that $\mu(C_n) \leqslant 2^{-n}$ and $N \subseteq \{x : \exists^\infty n \; x \in C_n\}$. Put $E_J = \{x : \forall^\infty m \; \exists n \in J_m \; x \in C_n\}$. Then $\langle E_J : J \in \mathrm{IP} \rangle$ is as desired. □

Partial result: proof

Partial result (G.)

GP(all) implies $\mathcal{SN} = \mathcal{NA}$.

Proof. Take $X \in \mathcal{SN}$ and $N \in \mathcal{N}$. We have to show $X + N \in \mathcal{N}$. By the lemma, we take a monotone family $\langle E_J : J \in \mathrm{IP} \rangle$ of members in $\mathcal{E}$ such that $N \subseteq \bigcup_{J \in \mathrm{IP}} E_J$. By the characterization of $\mathcal{SN}$, we have $X + E_J \in \mathcal{N}$ for every J. Then by GP(all), we have $\bigcup_{J \in \mathrm{IP}} (X + E_J) \in \mathcal{N}$. Thus, $X + N \in \mathcal{N}$. □

4

Hausdorff measure version of GP

Hausdorff measure: introduction

Hausdorff measures are measures that can finely measure Lebesgue null sets. Each Hausdorff measure is associated with a parameter $f \colon [0, \infty) \to [0, \infty)$ called a **gauge function**, that satisfies $f(0) = 0$ and is right-continuous and increasing. For a metric space (X, d), $A \subseteq X$ has ***f*-Hausdorff measure zero** iff

$$(\forall \varepsilon > 0)(\exists \langle C_n : n \in \omega \rangle \in \mathcal{P}(X)^\omega) \left[A \subseteq \bigcup_n C_n \wedge \sum_{n \in \omega} f(\mathrm{diam}(C_n)) < \varepsilon \right].$$

Hausdorff measure: result

Let $\mathrm{GP}(\Gamma, I)$ be the statement like $\mathrm{GP}(\Gamma)$ but the null ideal $\mathcal{N}$ is replaced by arbitrary ideal I.
Let $\mathcal{N}_X^f$ be the f-Hausdorff measure zero ideal on X.

Results (G.)

1. $\mathrm{GP}(\mathbf{\Sigma}^1_1, \mathcal{N}_X^f)$ holds when X is a metric space and f is a doubling gauge function.
2. The following statement is consistent: $\mathrm{GP}(\mathbf{\Pi}^1_1, \mathcal{N}_X^f)$ holds when X is a metric space and f is a doubling gauge function.

Here, that f is **doubling** means there is $r > 0$ such that for every $x > 0$ we have $f(2x) < rf(x)$.

Hausdorff measure: conjecture

Conjecture

The statement "GP(all, $\mathcal{N}_X^f$) holds for every metric space X and every doubling gauge function f" is consistent.

We suspect that this statement is also true in Laver model.

Partial result (G.)

Laver forcing preserves positivity of f-Hausdorff (outer) measure for every doubling gauge function f.

We need countable support iteration version of this result.

References

[Gol93] Martin Goldstern. "An Application of Shoenfield's Absoluteness Theorem to the Theory of Uniform Distribution.". In: *Monatshefte für Mathematik* 116.3-4 (1993), pp. 237–244.

[Got26] Tatsuya Goto. "Goldstern's priniciple about unions of null sets". In: *The Journal of Symbolic Logic, to apper* (2026).