Goldstern's principle for $\mathbf{\Pi}^1_1$ sets

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GOLDSTERN'S THEOREM

- In 1993, Martin Goldstern proved the following theorem:

THEOREM (Goldstern)

Let $\langle A_x : x \in \omega^\omega \rangle$ be a family of Lebesgue null sets indexed by ω^ω.

Assume the monotonicity condition: $\forall x, x' \in \omega^\omega (x \leq^* x' \Rightarrow A_x \subseteq A_{x'})$.

Assume also the set $\{(x, y) \in \omega^\omega \times 2^\omega : y \in A_x\}$ is $\mathbf{\Sigma}^1_1$.

Then the set $\bigcup_{x \in \omega^\omega} A_x$ is also Lebesgue null.

- He applied this theorem to problems in uniform distribution theory.

THE PRINCIPLE GP(I, Γ)

- We would like to generalize Goldstern's theorem.

DEFINITION

Let I be an ideal on $\mathbb{R}$ and Γ be a pointclass.

Then GP(I, Γ) is the following statement:

For every family $\langle A_x : x \in \omega^\omega \rangle$ of sets in I, <u>if</u> the following conditions hold:

(1) monotonicity condition: $\forall x, x' \in \omega^\omega (x \leq^* x' \Rightarrow A_x \subseteq A_{x'})$, and

(2) $\{(x, y) \in \omega^\omega \times 2^\omega : y \in A_x\}$ is in Γ,

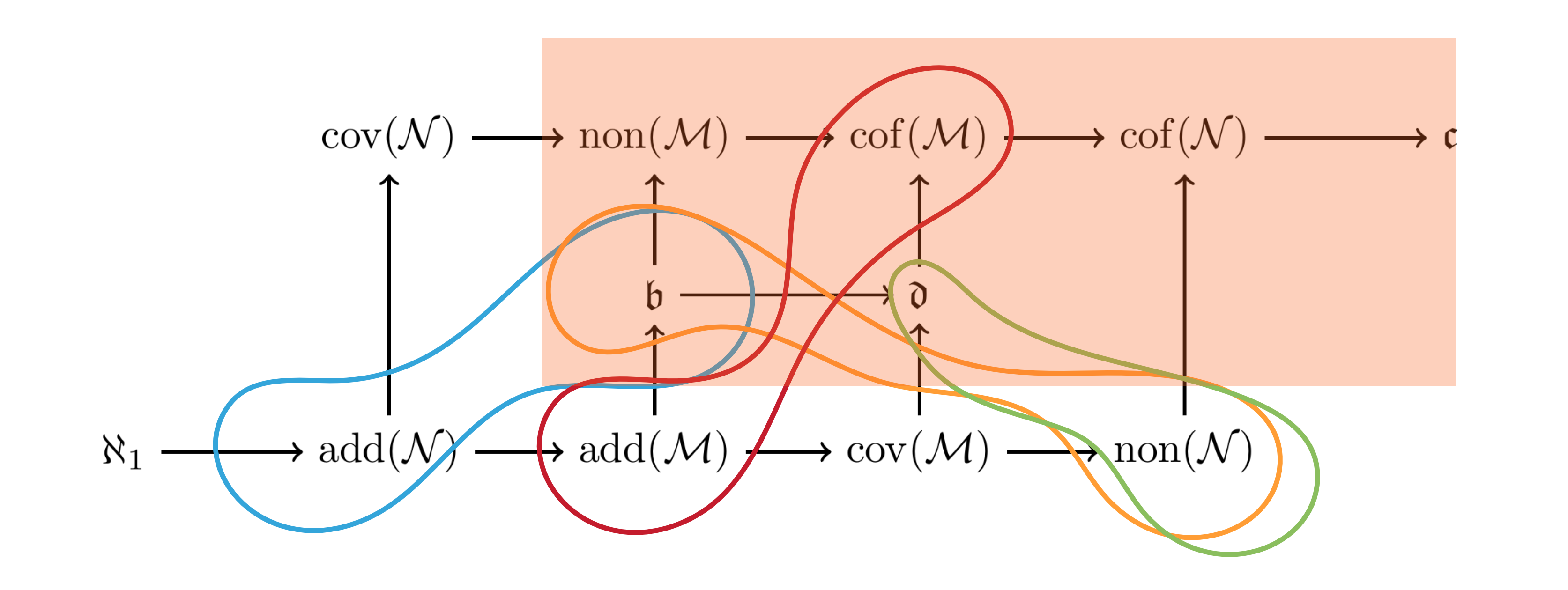
<u>then</u> the set $\bigcup_{x \in \omega^\omega} A_x$ is also in I.

- Goldstern's theorem = GP($\mathcal{N}, \Sigma^1_1$).

THE SPEAKER'S RESULTS

- In the preprint in 2022, the speaker showed the following theorems.
  - CH implies $\neg\mathrm{GP}(\mathcal{N}, \mathrm{all})$.
  - In Laver model, $\mathrm{GP}(\mathcal{N}, \mathrm{all})$ holds.
  - In L, $\neg\mathrm{GP}(\mathcal{N}, \mathbf{\Delta}^1_2)$ holds.
  - $\mathrm{ZF} + \mathrm{AD} \vdash \mathrm{GP}(\mathcal{N}, \mathrm{all})$.
  - In Solovay's model, $\mathrm{GP}(\mathcal{N}, \mathrm{all})$ holds.

The symbol "all" denotes the pointclass of all pointsets.



- This year, the speaker proved the following theorem.
  - $\mathrm{GP}(\mathcal{N}, \mathbf{\Pi}^1_1)$ holds.

PROOFS

$\mathrm{GP}(\mathcal{N}, \mathbf{\Sigma}^1_1)$

due to Goldstern

Using the random forcing and Shoenfield absoluteness

Key point: random forcing is ω^ω-bounding

$\mathrm{GP}(\mathcal{N}, \mathbf{\Pi}^1_1)$

due to the speaker

Using the Laver forcing and Shoenfield absoluteness

Key point: Laver forcing preserves the Lebesgue outer measure

Another ingredient: The Lebesgue measure of $\mathbf{\Sigma}^1_1$ sets can be calculated in Σ^1_1 manner (Kechris-Tanaka).

KECHRIS-TANAKA THEOREM

THEOREM (Kechris-Tanaka)

Let A be a $\boldsymbol{\Sigma}^1_1$ subset of $\omega^\omega \times 2^\omega$. Then the relation "$\mu(A_x) > r$" in the parameters x, r is $\boldsymbol{\Sigma}^1_1$.

- μ denotes the Lebesgue measure
- $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$

COROLLARY OF KECHRIS-TANAKA THEOREM

THEOREM (Kechris-Tanaka)

Let A be a $\mathbf{\Sigma}^1_1$ subset of $\omega^\omega \times 2^\omega$. Then the relation "$\mu(A_x) > r$" in the parameters x, r is $\mathbf{\Sigma}^1_1$.

COROLLARY

Let A be a $\mathbf{\Pi}^1_1$ subset of $\omega^\omega \times 2^\omega$. Then the relation "$\mu(A_x) = 0$" in the parameter x is $\mathbf{\Sigma}^1_1$.

Proof. Let B be the complement of A.

Then $\mu(A_x) = 0 \iff \mu(B_x) = 1 \iff (\forall n)[\mu(B_x) > 1 - 2^{-n}]$. ■

THE SPEAKER'S THEOREM

THEOREM (G.)

GP($\mathcal{N}, \mathbf{\Pi}^1_1$) holds.

Proof (1/2). Let $\langle A_x : x \in \omega^\omega \rangle$ be a family of Lebesgue null sets satisfying the monotonicity condition and $\mathbf{\Pi}^1_1$ condition.

Let ℓ be the Laver real over the ground model V.

The statement $(\forall x \in \omega^\omega)(\mu(A_x) = 0)$ is true in V and it can be written in $\mathbf{\Pi}^1_2$ sentence using the corollary of Kechris-Tanaka theorem.

So by Shoenfield's absoluteness, it holds also in $V[\ell]$.

Thus, $V[\ell]$ satisfies $\mu(A_\ell) = 0$.

THE SPEAKER'S THEOREM

THEOREM (G.)

$\mathrm{GP}(\mathcal{N}, \mathbf{\Pi}^1_1)$ holds.

Proof (2/2). Also, the monotonicity condition: $\forall x, x' \in \omega^\omega (x \leq^* x' \Rightarrow A_x \subseteq A_{x'})$ can be written in $\mathbf{\Pi}^1_2$ sentence. So it holds in $V[\ell]$ since it holds in V.

Since the Laver real ℓ is a dominating real over V, it holds that $\bigcup_{x \in \omega^\omega \cap V} A_x \subseteq A_\ell$.

But, since A is $\mathbf{\Pi}^1_1$, it holds also that $(\bigcup_{x \in \omega^\omega} A_x)^V \subseteq \bigcup_{x \in \omega^\omega \cap V} A_x$. So $(\bigcup_{x \in \omega^\omega} A_x)^V$ is null in $V[\ell]$.

On the other hand, we know the Laver forcing preserves Lebesgue outer measure. Therefore, $(\bigcup_{x \in \omega^\omega} A_x)^V$ is null also in V. ■

THE FUTURE STUDY:

How about the case when the ideal I is not the Lebesuge null ideal $\mathcal{N}$?

HAUSDORFF MEASURES

- Hausdorff measures are measures that can finely measure Lebesgue null sets.
- Each Hausdorff measure is associated with a parameter $f : [0,\infty) \to [0,\infty)$ called a gauge function, that satisfies $f(0) = 0$ and f is right-continuous and increasing.
- For a metric space (X, d), $A \subseteq X$ has f-Hausdorff measure zero
- $$\text{iff } (\forall \epsilon > 0)(\exists \langle C_n : n \in \omega \rangle \in \mathscr{P}(X)^{\omega}) \left[A \subseteq \bigcup_n C_n \wedge \sum_{n \in \omega} f(\mathrm{diam}(C_n)) < \varepsilon \right]$$

	$I = \mathcal{N}$	I is the f-Hausdorff measure zero ideal, where f is a doubling gauge.	I is the f-Hausdorff measure zero ideal, where f is a non-doubling gauge.
$\mathrm{GP}(I, \mathbf{\Sigma}^1_1)$	TRUE	TRUE	UNKNOWN
$\mathrm{GP}(I, \mathbf{\Pi}^1_1)$	TRUE	UNKNOWN	UNKNOWN
$\mathrm{Con}(\mathrm{ZFC} + \mathrm{GP}(I, \mathrm{all}))$	TRUE	UNKNOWN	UNKNOWN
$\mathrm{ZF} + \mathrm{AD} \vdash \mathrm{GP}(I, \mathrm{all})$	TRUE	TRUE	UNKNOWN

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