Games related to splitting families

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3. Splitting* games
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    - An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$
4. Conclusion

Game-theoretic variants of cardinal invariants

The study of cardinal invariants of the continuum is important in set theory of reals. On the other hand, the study of infinite games is also an important topic in set theory.

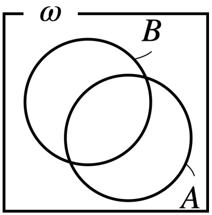
Our study connects the two fields of cardinal invariants and game theory by examining what can be obtained from game-theoretic modifications of cardinal invariants.

We don't assume the axiom of determinacy in this presentation. But we assume the axiom of choice.

Summary of our results

game	$\mathfrak{x}^{\mathrm{I}}_{\mathrm{game}}$	$\mathfrak{x}^{\mathrm{II}}_{\mathrm{game}}$
splitting	$\mathbf{s}_\sigma$	$\mathbf{c}$
splitting*	$\mathbf{s}_\sigma \leq\ ?\ \leq \min\{\mathrm{non}(\mathcal{M}), \mathbf{d}, \mathrm{non}(\mathcal{N})\}$	$\mathbf{c}$
reaping	$\max\{\mathbf{r}, \mathbf{d}\} \leq\ ?\ \leq \max\{\mathbf{r}_\sigma, \mathbf{d}\}$	$\mathbf{c}$
reaping*	∞	∞
bounding	$\mathbf{b}$	$\mathbf{d}$
bounding*	$\mathbf{b}$	$\mathbf{c}$
dominating	$\mathbf{d}$	$\mathbf{d}$
dominating*	$\mathbf{d}$	$\mathbf{c}$
anti-localizing	$\mathrm{add}(\mathcal{N})$	$\mathrm{cov}(\mathcal{M})$
anti-localizing*	$\mathrm{add}(\mathcal{N})$	$\mathbf{c}$

The definition of the splitting number



For infinite subsets A, B of ω, we say **A splits B** if

$$|B \cap A| = |B \setminus A| = \aleph_0.$$

For $\mathcal{S} \subseteq [\omega]^\omega$, we say

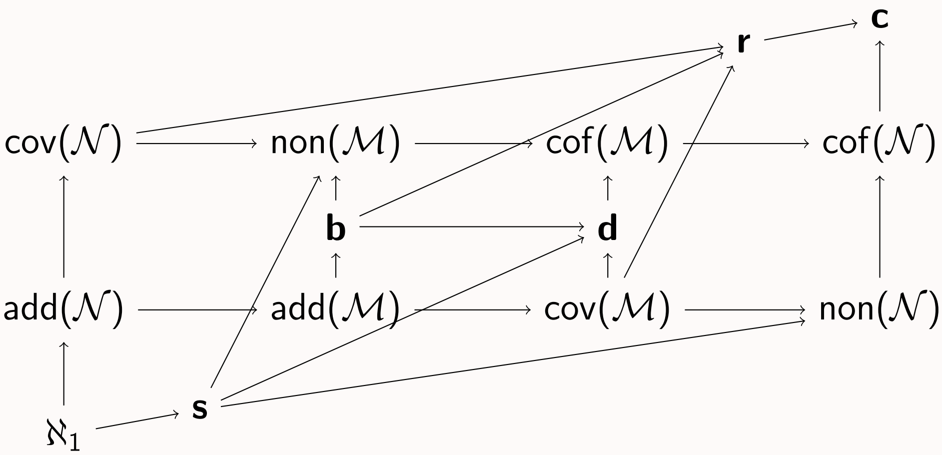
- $\mathcal{S}$ is a **splitting family**
$$:\iff (\forall B \in [\omega]^\omega)(\exists A \in \mathcal{S})(A \text{ splits } B).$$

The cardinal $\mathbf{s}$ defined below is called the splitting number:

- $\mathbf{s} := \min\{|\mathcal{S}| : \mathcal{S} \text{ is a splitting family}\}.$

s and cardinal invariants

s is a typical example of cardinal invariants of the continuum.



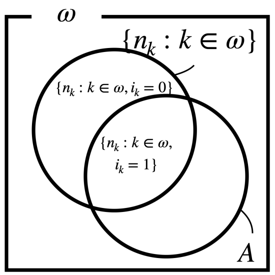
Splitting games

Fix $\mathcal{A} \subseteq [\omega]^\omega$. We call the following game the **splitting game** with respect to $\mathcal{A}$:

Player I	n_0	$<$	n_1	$<$	$\dots$	
Player II		$i_0 \in 2$		$i_1 \in 2$		$\dots$

Player II wins $\Leftrightarrow$ Player II played both 0 and 1 infinitely and there is $A \in \mathcal{A}$ such that

$$\{n_k : k \in \omega\} \cap A = \{n_k : k \in \omega \text{ and } i_k = 1\}.$$



Movie

Cardinal invariants on splitting games

Definition

$$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega),\ \text{In the splitting game with respect to } \mathcal{A},\ \text{Player I does not have a winning strategy}\}$$

$$\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega),\ \text{In the splitting game with respect to } \mathcal{A},\ \text{Player II has a winning strategy}\}$$

Theorems about splitting games

We can easily see the following.

Proposition (Chapital–G.–Hayashi)

$\mathbf{s} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{II}}_{\mathrm{game}} \leq \mathbf{c}$.

The following needs some discussion.

Theorem (Chapital–G.–Hayashi)

$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} = \mathbf{s}_\sigma$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} = \mathbf{c}$.

(We will see the definition of $\mathbf{s}_\sigma$ in the next page.)

Definition of σ-splitting number

For $A \in [\omega]^\omega$ and $f : \omega \to [\omega]^\omega$, we say A σ-splits f if

$$\text{For every } n,\ A \text{ splits } f(n).$$

For $\mathcal{S} \subseteq [\omega]^\omega$, we say $\mathcal{S}$ is a **σ-splitting family**

$$:\iff (\forall f : \omega \to [\omega]^\omega)(\exists A \in \mathcal{S})(A\ \sigma\text{-splits } f).$$

We call the following cardinal the σ-splitting number:

- $\mathbf{s}_\sigma := \min\{|\mathcal{S}| : \mathcal{S} \text{ is a } \sigma\text{-splitting family}\}$.

It can be easily seen that $\mathbf{s} \leq \mathbf{s}_\sigma$.

Note It is a longstanding open question whether ZFC proves $\mathbf{s} = \mathbf{s}_\sigma$!

The proof of $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$ (1/2)

Theorem $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$.

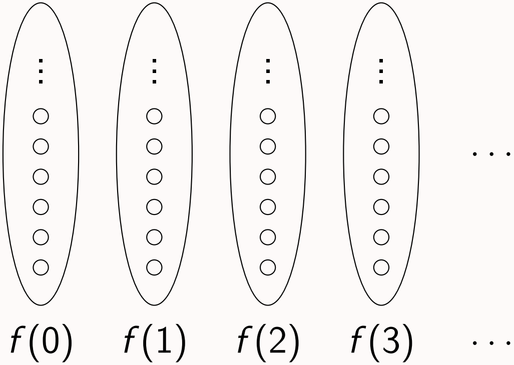
Proof. Fix a family $\mathcal{A} \subseteq [\omega]^\omega$ such that Player I has no winning strategy for the splitting game with respect to $\mathcal{A}$. We want to show that $\mathcal{A}$ is a σ-splitting family.

Take $f : \omega \to [\omega]^\omega$. We shall find $A \in \mathcal{A}$ such that A splits $f(n)$ for every $n \in \omega$.

The proof of $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$ (2/2)

Theorem $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$.

Consider the following strategy of Player I.



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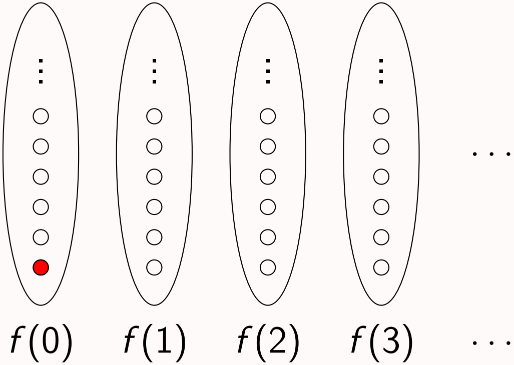

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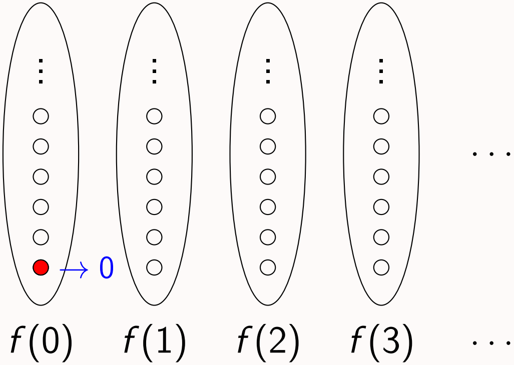

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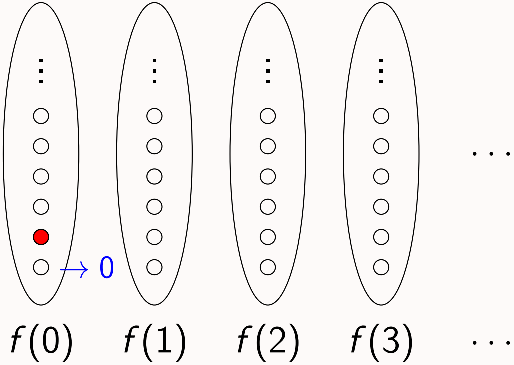

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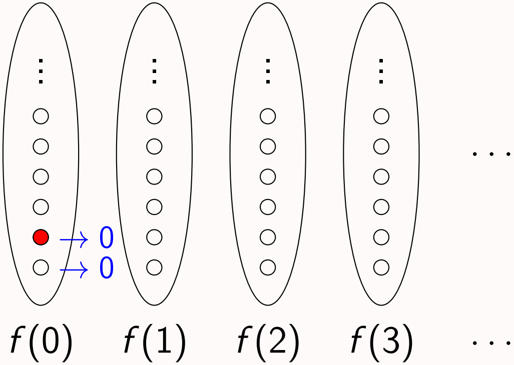

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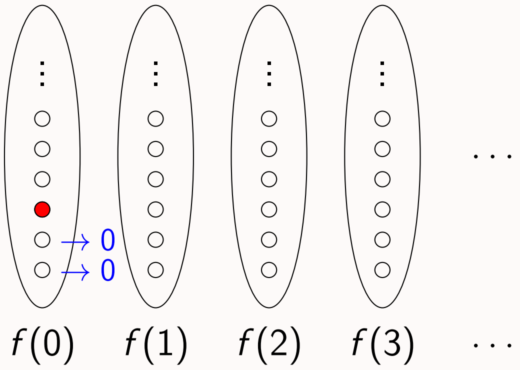

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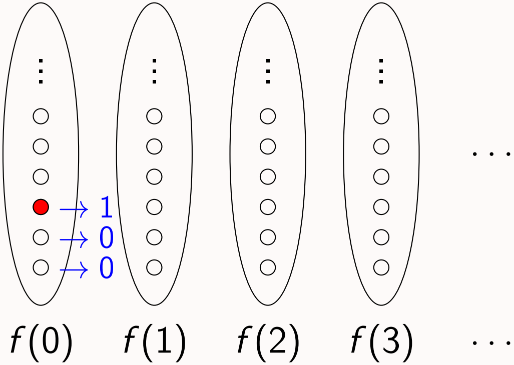

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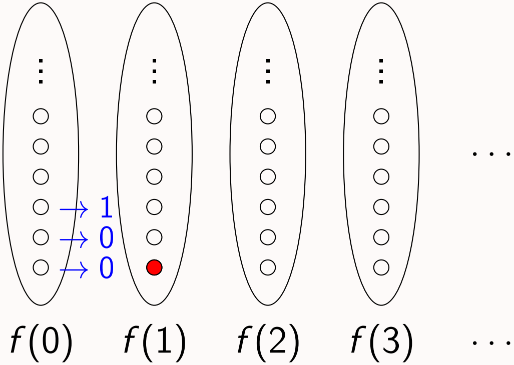

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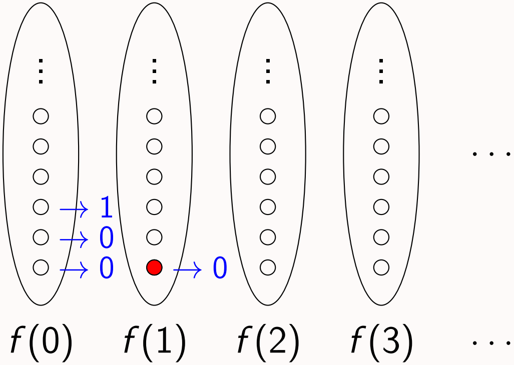

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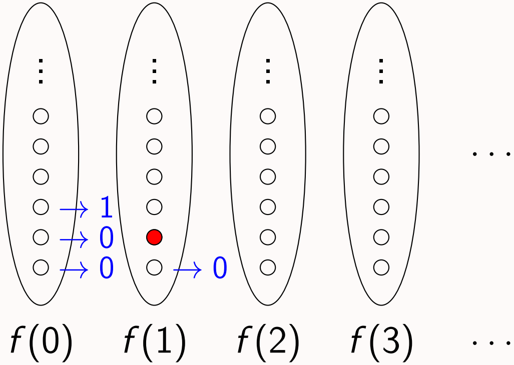

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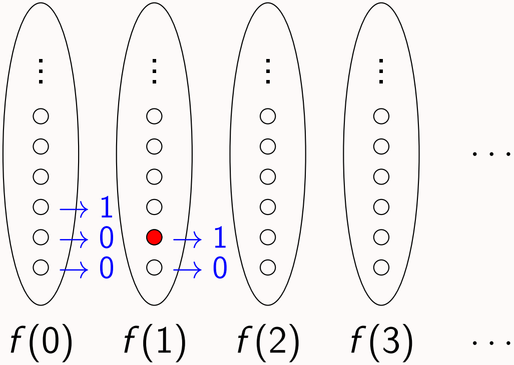

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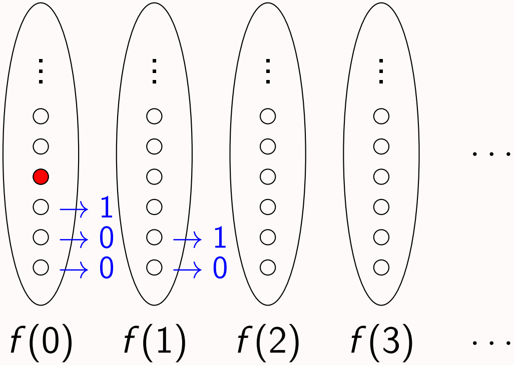

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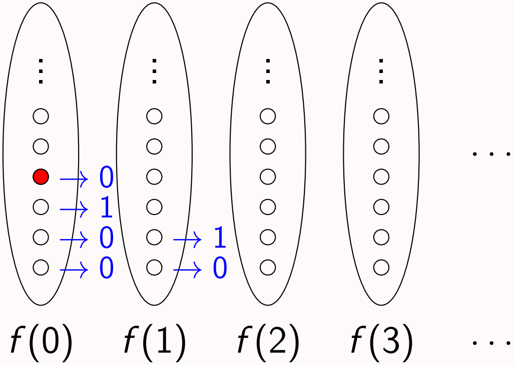

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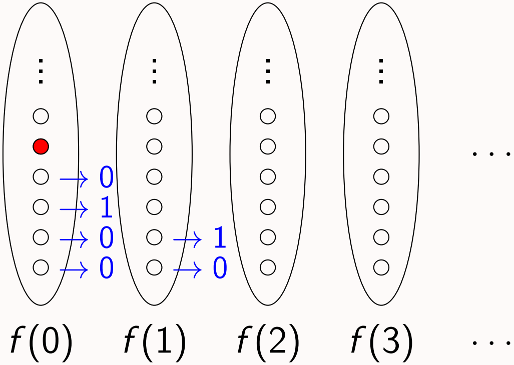

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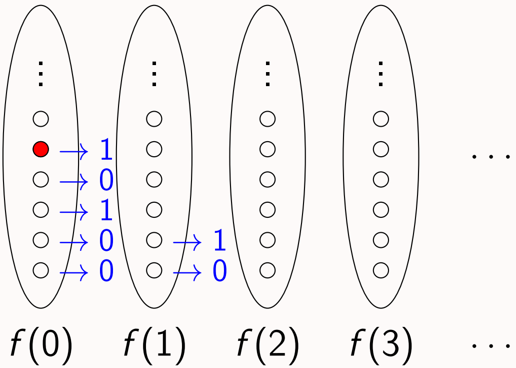

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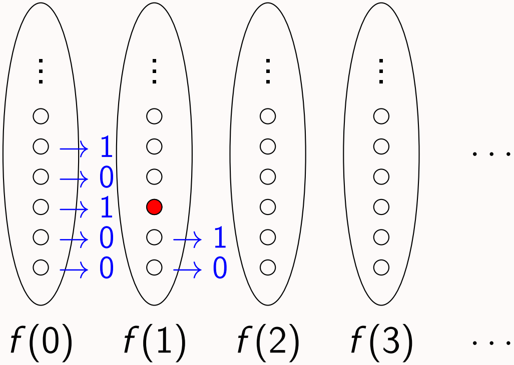

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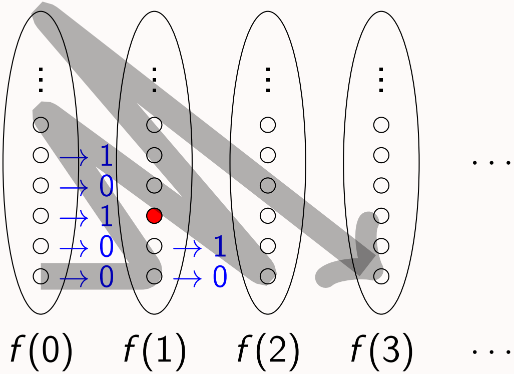


splitting* game

Fix $\mathcal{A} \subseteq [\omega]^\omega$. We call the following game **splitting* game** with respect to $\mathcal{A}$.

Player I	$i_0 \in 2$		$i_1 \in 2$		$\dots$	
Player II		$j_0 \in 2$		$j_1 \in 2$		$\dots$

Player II wins if either Player I said 1 finitely or

$$\{k \in \omega : j_k = 1\} \text{ is in } \mathcal{A} \text{ and splits } \{k \in \omega : i_k = 1\}.$$

Cardinal invariants on splitting* games

Definition

$$\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting* game with respect to } \mathcal{A}, \text{ Player I does not have a winning strategy}\}$$

$$\mathfrak{s}^{\mathrm{II}}_{\mathrm{game}^*} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting* game with respect to } \mathcal{A}, \text{ Player II has a winning strategy}\}$$

An observation of splitting* games

The splitting* game with respect to $\mathcal{A}$ is a harder game for Player II than the splitting game with respect to $\mathcal{A}$.

Therefore, $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{II}}_{\mathrm{game}^*}$ hold.

Thus, we have $\mathbf{s}_{\sigma} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}^*} = \mathbf{c}$.

Theorems about splitting* games

Theorem (Chapital–G.–Hayashi)

The proposition $\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ is relatively consistent from ZFC.

Theorem (Chapital–G.–Hayashi)

$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{M}), \mathbf{d}, \mathrm{non}(\mathcal{N})$.

Consistency proof

Theorem Con($\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$).

We use the following fact.

Fact by Judah–Shelah Assume CH. Then every finite support iteration of Suslin ccc forcing forces that $\mathbf{s} = \aleph_1$.

Using this theorem, it is enough to show that there is a Suslin ccc forcing that adds a generic strategy of Player I that wins every play in the ground model.

Consistency proof

Theorem Con($\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$).

Define such a forcing poset P as follows:

$$P = \{(n, s, H) : n \in \omega, s\colon 2^{<n} \to 2, H \subseteq 2^\omega \setminus \mathbb{O} \text{ finite}\}.$$

Here, $\mathbb{O}$ is the set of eventually zero sequences. The order is:

$$(n', s', H') \leq (n, s, H) \iff n \leq n', s \subseteq s', H \subseteq H' \text{ and}$$
$$(\forall x \in H)(\forall i \in [n, n'))(x(i) = 0 \to s'(x \restriction i) = 0).$$

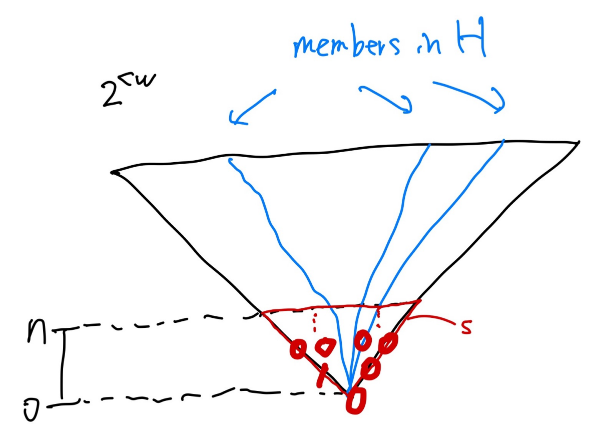
Define a P-name $\dot\sigma$ as follows:

$$\Vdash \dot\sigma = \bigcup\{s : (n, s, H) \in G\}.$$

This $\dot\sigma$ is the desired name for the generic strategy.

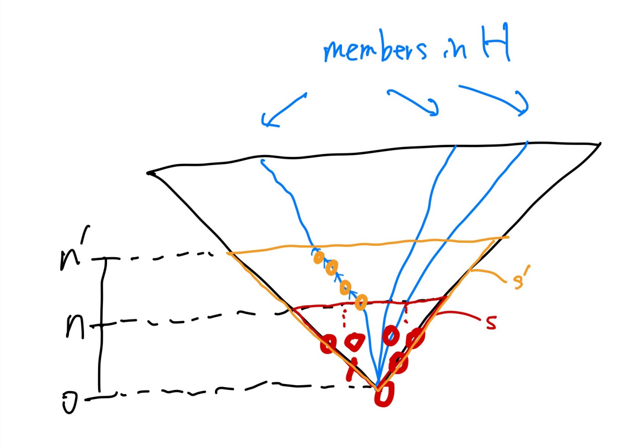
Consistency proof

Theorem Con($\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$).



Consistency proof

Theorem $\mathrm{Con}(\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*})$.



Consistency proof

Theorem Con($\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$).

By density arguments, we can show the following:

1. $\Vdash (\forall x \in (2^\omega \setminus \mathbb{O}) \cap V)(\{k \in \omega : \dot{\sigma}(x \restriction k) = 1\} \subseteq^* x^{-1}\{1\})$.
2. $\Vdash (\forall x \in 2^\omega \cap V)(\{k \in \omega : \dot{\sigma}(x \restriction k) = 1\}$ is infinite$)$.

In the case $x \in \mathbb{O}$, it is clear that

$$\Vdash \{k \in \omega : \dot{\sigma}(x \restriction k) = 1\} \subseteq^* x^{-1}\{0\}.$$

□

An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Recall that
$\mathrm{non}(\mathcal{N}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq 2^\omega, \mathcal{A} \text{ is a Lebesgue non-null set}\}$.

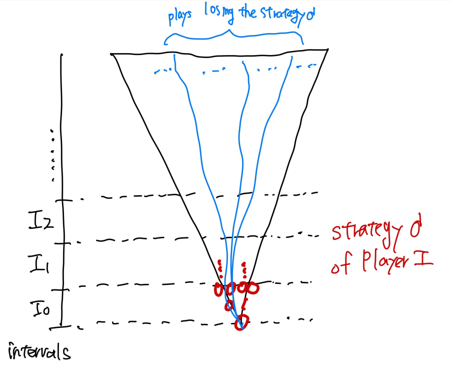
An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Plan. Fixing the strategy σ of Player I, it is sufficient to say that the following set has measure 0:

$$\{x \in 2^\omega : \text{the strategy } \sigma \text{ wins the play } x \text{ in splitting* game}\}$$

Let us divide this set into intervals, and then reduce the problem to counting up finite sets.



An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Lemma 1 Let $I = [i, j)$ be an interval in ω. Let $s \in \{0,1\}^i$, $\sigma \colon \{0,1\}^{<j} \to 2$ and $\varepsilon \in 2$. Set

$$B^I_{s,\varepsilon}(\sigma) = \{x \in \{0,1\}^j : s \subseteq x, (\exists k \in I)(\sigma(x \restriction k) = 1), \text{ and } (\forall k \in I)(\sigma(x \restriction k) = 1 \to x(k) = \varepsilon)\}.$$

Then we have $\dfrac{|B^I_{s,\varepsilon}(\sigma)|}{2^{j-i}} \leq \dfrac{1}{2}$.

(Proof) Induction on $|I|$. □

An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Lemma 2 Let $a < b < \omega$. Let $\bar{I} = \langle I_n : a \leq n < b \rangle$ be a sequence of consecutive intervals in ω and put $m := \min I_a$ and $M := \max I_{b-1} + 1$. Let $\sigma \colon \{0,1\}^{<M} \to 2$ and $\varepsilon \in 2$. Set

$$B^{\bar{I}}_{\varepsilon}(\sigma) = \{x \in \{0,1\}^M : (\forall n \in [a,b))[(\exists k \in I_n)(\sigma(x \restriction k) = 1), \text{ and } (\forall k \in I_n)(\sigma(x \restriction k) = 1 \to x(k) = \varepsilon)]\}.$$

Then we have $\dfrac{|B^{\bar{I}}_{\varepsilon}(\sigma)|}{2^M} \leq \dfrac{1}{2^{b-a}}$.

(Proof) Use Lemma 1 repeatedly. □

An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Let IP be the set of all interval partitions of ω. For $\bar{I}, \bar{J} \in \mathrm{IP}$, we define

$$\bar{I} \leq^* \bar{J} :\Leftrightarrow (\forall^\infty m)(\exists n)(I_n \subseteq J_m).$$

We use the following theorem due to Martin Goldstern.

Fact by Goldstern Let $A \subseteq \mathrm{IP} \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Suppose that the vertical section $A_{\bar{I}}$ is null for every $\bar{I} \in \mathrm{IP}$ and $A_{\bar{I}} \subseteq A_{\bar{J}}$ for every $\bar{I}, \bar{J} \in \mathrm{IP}$ with $\bar{I} \leq^* \bar{J}$. Then $\bigcup_{\bar{I} \in \mathrm{IP}} A_{\bar{I}}$ is null.

An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be a non-null set of size $\mathrm{non}(\mathcal{N})$. We will show that Player I has no winning strategy for the splitting* game with respect to $\mathcal{A}$. Fix a strategy $\sigma \colon 2^{<\omega} \to 2$ of Player I.
For $\bar{I} \in \mathrm{IP}$ and $\varepsilon \in 2$, define

$$C^{\bar{I}}_{\varepsilon} = \bigcup_{a\in\omega} \bigcap_{b>a} B^{\bar{I}\restriction[a,b)}_{\varepsilon}(\sigma).$$

By Lemma 2, this set $C^{\bar{I}}_{\varepsilon}$ is null.

An upper bound of $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$

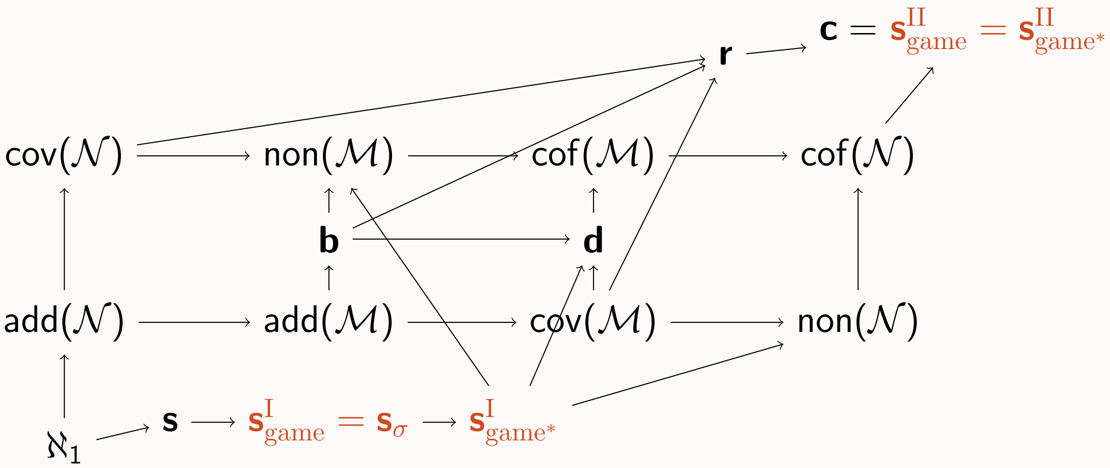
Theorem $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$.

Moreover when $\bar{I} \leq^* \bar{J}$, we have $C^{\bar{I}}_{\varepsilon} \subseteq C^{\bar{J}}_{\varepsilon}$. Also the set $\{(\bar{I}, x) : x \in C^{\bar{I}}_{\varepsilon}\}$ is clearly a Borel set. Therefore, we can apply Goldstern's theorem to get that $\bigcup_{\bar{I} \in \mathrm{IP}} C^{\bar{I}}_{\varepsilon}$ is null.
Moreover we can easily observe that

$$\{x \in 2^{\omega} : \text{the strategy } \sigma \text{ wins the play } x \text{ in splitting* game}\}$$
$$\subseteq \bigcup_{\bar{I} \in \mathrm{IP}} C^{\bar{I}}_{0} \cup \bigcup_{\bar{I} \in \mathrm{IP}} C^{\bar{I}}_{1}.$$

So we can take $x \in \mathcal{A}$ that avoids this set. This means σ is not a winning strategy for the splitting* game with respect to $\mathcal{A}$. □

The diagram with invariants regarding splitting games



Quesitions

- In the model where we proved $\mathsf{s} < \mathsf{s}^{\mathrm{I}}_{\mathrm{game}^*}$, does it hold that $\mathsf{s}_\sigma = \aleph_1$?
- Can we show $\mathsf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{E})$ in ZFC?

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