Keisler's Theorem and Cardinal Invariants

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Abstract

Keisler–Shelah isomorphism theorem states that two models are elementarily equivalent if and only if their ultrapowers with respect to some ultrafilter over some set are isomorphic.

Especially, Keisler's theorem states that under CH, an ultrafilter over ω witnesses the above statement if the languages are countable and the cardinalities of the structures are $\leq \mathfrak{c}$.

We discuss relations between Keisler's theorem and cardinal invariants.

Notations

- $\mathfrak{c} := 2^{\aleph_0}$.
- $\mathcal{N}$ denotes the null ideal.
- $\mathcal{M}$ denotes the meager ideal.

Review of saturability

Let $\mathcal{L}$ be a (first-order) language and $\mathcal{A}$ be an $\mathcal{L}$-structure. Let p be a set of $\mathcal{L}(\mathcal{A})$-formulas with one fixed free variable x. We say p is **finitely satisfiable** if for every finite subset Σ of p there exists $x \in \mathcal{A}$ that satisfies all formulas in Σ. For $x \in \mathcal{A}$, we say x **realizes** p if x satisfies all formulas in p.

For a cardinal κ, we say $\mathcal{A}$ is **κ-saturated** if for every finitely satisfiable set p of $\mathcal{L}(\mathcal{A})$-formulas with the number of parameters occuring in p being $< \kappa$, there is an element of $\mathcal{A}$ that realizes p.

Definitions of KT and SAT

Let κ be a cardinal.

We say KT(κ) **holds** if for every countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \kappa$ with $\mathcal{A} \equiv \mathcal{B}$, there exists an ultrafilter U over ω such that $\mathcal{A}^\omega/U \simeq \mathcal{B}^\omega/U$.

We say SAT(κ) **holds** if there exists an ultrafilter U over ω such that for every language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leq \kappa$, $\prod_{i\in\omega} \mathcal{A}_i/U$ is finite or $\mathfrak{c}$-saturated.

Known results

We say **KT(κ) holds** if for every countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \kappa$ with $\mathcal{A} \equiv \mathcal{B}$, there exists an ultrafilter U over ω such that $\mathcal{A}^\omega/U \simeq \mathcal{B}^\omega/U$.

We say **SAT(κ) holds** if there exists an ultrafilter U over ω such that for every language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leq \kappa$, $\prod_{i\in\omega} \mathcal{A}_i/U$ is finite or $\mathfrak{c}$-saturated.

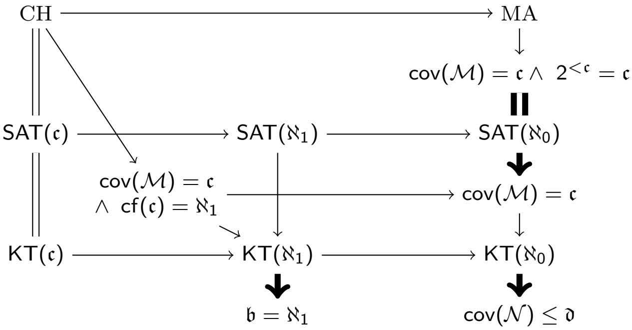
1. SAT(κ) implies KT(κ) for every κ.
2. (Keisler, 1961) CH implies SAT($\mathfrak{c}$).
3. (Ellentuck–Rucker, 1972) MA implies SAT($\aleph_0$).
4. (Shelah, 1992) KT($\aleph_0$) implies $\mathfrak{v}^\forall \leq \mathfrak{d}$.
5. (Golshani–Shelah, 2021) $\neg$KT($\aleph_2$). In particular, CH iff KT($\mathfrak{c}$).
6. (Golshani–Shelah, 2021) $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) = \aleph_1$ implies KT($\aleph_1$).
7. (Golshani–Shelah, 2021) In the Cohen model, KT($\aleph_1$) holds.

Diagram of implications

We say KT(κ) **holds** if for every countable language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \kappa$ with $\mathcal{A} \equiv \mathcal{B}$, there exists an ultrafilter U over ω such that $\mathcal{A}^\omega/U \simeq \mathcal{B}^\omega/U$.

We say SAT(κ) **holds** if there exists an ultrafilter U over ω such that for every language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leq \kappa$, $\prod_{i\in\omega} \mathcal{A}_i/U$ is finite or $\mathfrak{c}$-saturated.

Thick arrows indicate our results.



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$\neg\,\mathrm{KT}(\aleph_2)$

- Toward a contradiction, assume $\mathrm{KT}(\aleph_2)$.
- Define a language $\mathcal{L}$ by $\mathcal{L} = \{<\}$ and put $\mathcal{A} = (\mathbb{Q}, <)$, $\mathcal{B} = (\mathbb{Q} + (\omega_2 + 1) \times \mathbb{Q}_{\geq 0}, <_{\mathcal{B}})$. Here $<_{\mathcal{B}}$ is defined by the lexicographical order.
- We have $|\mathcal{A}| = \aleph_0, |\mathcal{B}| = \aleph_2$.
- $\mathcal{A}, \mathcal{B}$ are both dense linear ordered sets. So by completeness of DLO, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent.
- Then by $\mathrm{KT}(\aleph_2)$, we can take U such that $\mathcal{B}^\omega/U \simeq \mathcal{A}^\omega/U$.
- Put $\mathcal{A}^* = \mathcal{A}^\omega/U, \mathcal{B}^* = \mathcal{B}^\omega/U$.

$\neg\,\mathsf{KT}(\aleph_2)$

The idea of proof is that $\mathbb{Q}$ is "homogeneous" and $\mathcal{B}$ is "rugged" and these properties are inherited by their ultrapowers.

$\neg\,\mathsf{KT}(\aleph_2)$

- Take $a, b \in \mathcal{B}$ such that $\mathrm{cf}(\mathcal{B}_a) = \omega_1, \mathrm{cf}(\mathcal{B}_b) = \omega_2$. Here

$$\mathcal{B}_c = \{d \in \mathcal{B} : d <_{\mathcal{B}} c\}.$$

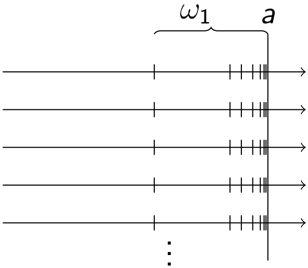
- Put $a_* = [\langle a, a, a, \dots \rangle], b_* = [\langle b, b, b, \dots \rangle] \in \mathcal{B}^*$.

$\neg\,\mathrm{KT}(\aleph_2)$

Lemma

$\mathrm{cf}((\mathcal{B}^*)_{a_*}) = \omega_1, \mathrm{cf}((\mathcal{B}^*)_{b_*}) = \omega_2$.

$\because$ By $\mathrm{cf}(\mathcal{B}_a) = \omega_1$, take an increasing cofinal sequence $\langle a_i : i < \omega_1 \rangle$. Then $\langle a_i^* : i < \omega_1 \rangle$, where $a_i^* = [\langle a_i, a_i, a_i, \ldots \rangle]_U$, is a cofinal sequence of $(\mathcal{B}^*)_{a_*}$ (by regularity of ω_1). Thus $\mathrm{cf}((\mathcal{B}^*)_{a_*}) = \omega_1$. The proof for $\mathrm{cf}((\mathcal{B}^*)_{b_*}) = \omega_2$ is similar. //



$\neg\,\mathsf{KT}(\aleph_2)$

Lemma

There is a function $F\colon \mathbb{Q}^3 \to \mathbb{Q}$ such that for any $c, d \in \mathbb{Q}$, the function $x \mapsto F(x, c, d)$ is an automorphism on $(\mathbb{Q}, <)$ that sends c to d.

$\because$ $F(x, y, z) = x - y + z$ suffices. //

- Now consider the function F_* from $(\mathcal{A}^*)^3$ to $\mathcal{A}^*$ induced by F. Then we have:

$$(*) \quad \begin{array}{l} F_*\colon (\mathcal{A}^*)^3 \to \mathcal{A}^* \text{ satisfies for any } c, d \in \mathcal{A}^*, \\ x \mapsto F_*(x, c, d) \text{ is an automorphism on } \mathcal{A}^* \\ \text{that sends } c \text{ to } d. \end{array}$$

- Therefore in $\mathcal{A}^*$, for every two points c, d, we have $\mathrm{cf}((\mathcal{A}^*)_c) = \mathrm{cf}((\mathcal{A}^*)_d)$.
- So $\mathcal{A}^*$ and $\mathcal{B}^*$ are not isomorphic. □

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$\mathrm{KT}(\aleph_1)$ implies $\mathfrak{b} = \aleph_1$

- Assume $\mathrm{KT}(\aleph_1)$.
- Define a language $\mathcal{L}$ by $\mathcal{L} = \{<\}$ and put $\mathcal{A} = (\mathbb{Q}, <)$, $\mathcal{B} = (\mathbb{Q} + (\omega_1 + 1) \times \mathbb{Q}_{\geq 0}, <_{\mathcal{B}})$.
- Then by $\mathrm{KT}(\aleph_1)$, we can take U such that $\mathcal{B}^\omega/U \simeq \mathcal{A}^\omega/U$.
- Put $\mathcal{A}^* = \mathcal{A}^\omega/U, \mathcal{B}^* = \mathcal{B}^\omega/U$.

KT($\aleph_1$) implies $\mathfrak{b} = \aleph_1$

- By the same reason as in the proof of $\neg KT(\aleph_2)$, We have the following observation:

  A point with cofinality ω_1 remains to have cofinality ω_1 in the ultrapower.

- On the other hand, a point with cofinality ω increases its cofinality in the ultrapower to $\mathrm{cf}(\omega^\omega/U, <_U)$. We can see this by mapping a sequence rapidly converging to the point into rapidly increasing function in ω^ω.
- $\mathrm{cf}(\omega^\omega/U, <_U) \geq \mathfrak{b}$.
- In $\mathcal{B}^*$, cofinalities of all points are same, we have $\mathfrak{b} = \aleph_1$. $\square$

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Definition of (anti-)localization cardinals

Let $c, h \in \omega^\omega$. We define

$$\prod c = \prod_{i \in \omega} c(i)$$

$$S(c, h) = \prod_{i \in \omega} [c(i)]^{\leq h(i)}$$

Definition of (anti-)localization cardinals

For $c, h \in \omega^\omega$, define

$$\mathfrak{c}^{\forall}_{c,h} = \min\{|S| : S \subseteq S(c,h), (\forall x \in \prod c)(\exists \varphi \in S)(\forall^\infty n)(x(n) \in \varphi(n))\}$$

$$\mathfrak{c}^{\exists}_{c,h} = \min\{|S| : S \subseteq S(c,h), (\forall x \in \prod c)(\exists \varphi \in S)(\exists^\infty n)(x(n) \in \varphi(n))\}$$

$$\mathfrak{v}^{\forall}_{c,h} = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)(\exists^\infty n)(x(n) \notin \varphi(n))\}$$

$$\mathfrak{v}^{\exists}_{c,h} = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)(\forall^\infty n)(x(n) \notin \varphi(n))\}$$

Definition of (anti-)localization cardinals

Put

$$\mathfrak{v}^{\forall} = \min\{\mathfrak{v}^{\forall}_{c,h} : c, h \in \omega^{\omega}, \lim_{i \to \infty} h(i) = \infty\}.$$

and put

$$\mathfrak{c}^{\exists} = \min\{\mathfrak{c}^{\exists}_{c,h} : c, h \in \omega^{\omega}, \sum_{i \in \omega} h(i)/c(i) < \infty\}.$$

Fact from Klausner–Mejía [KM19]

$\mathrm{cov}(\mathcal{N}) \leq \mathfrak{c}^{\exists}$ and $\mathfrak{v}^{\forall} \leq \mathfrak{c}^{\exists}$.

Shelah's result and our improvement

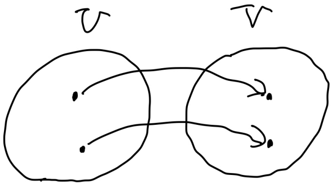
In [She92], Shelah proved $\mathrm{KT}(\aleph_0)$ implies $\mathfrak{v}^\forall \leq \mathfrak{d}$. We showed that $\mathrm{KT}(\aleph_0)$ implies $\mathfrak{c}^\exists \leq \mathfrak{d}$. This is an improvement of Shelah's result because of Fact in the previous page.

Since $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{c}^\exists$, in the random model, $\mathfrak{c}^\exists = \mathfrak{c}$ while $\mathfrak{d} = \aleph_1$. Thus in the model $\neg(\mathfrak{c}^\exists \leq \mathfrak{d})$ holds. So the consistency of $\neg\,\mathrm{KT}(\aleph_0)$ can be obtained by the random model.

Review of Shelah's construction of models

Define a language $\mathcal{L}$ by $\mathcal{L} = \{E, U, V\}$, where E is a binary predicate and U, V are unary predicates. We say an $\mathcal{L}$-structure $M = (|M|, E^M, U^M, V^M)$ is a **bipartite graph** if the following conditions hold:

1. $U^M \cup V^M = |M|$,
2. $U^M \cap V^M = \varnothing$,
3. $(\forall x, y \in |M|)(x\, E^M\, y \to (x \in U^M \wedge y \in V^M))$.



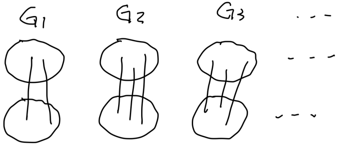
Review of Shelah's construction of models

For $n, k \in \omega$, define a bipartite graph $\Delta_{n,k}$ as follows:

1. $U^{\Delta_{n,k}} = \{1, 2, \ldots, n\}$
2. $V^{\Delta_{n,k}} = [\{1, 2, \ldots, n\}]^{\leq k} \setminus \{\varnothing\}$
3. For $u \in U^{\Delta_{n,k}}, v \in V^{\Delta_{n,k}}$, $u \, E^{\Delta_{n,k}} \, v$ iff $u \in v$.

For $n \in \omega$, Let $G_n = \Delta_{n^2+1,n}$. Let Γ be the bipartite graph obtained by taking the disjoint union of $\langle G_n : n \geq 1 \rangle$.

We can define a natural order on Γ by $x \lhd y$ if $m < n$ for $x \in G_m, y \in G_n$. Then Γ is a bipartite graph with an order $\lhd$.



Review of Shelah's construction of models

Put $\mathcal{L}' = \mathcal{L} \cup \{\triangleleft\}$. We consider $\mathcal{L}'$-structures which are elementarily equivalent to Γ.

Let Γ_{NS} be a countable proper elementary extension of Γ.

Review of Shelah's construction of models

When we say connected components, we mean the connected components when we ignore the orientation of the edges.

Lemma

Let $\mathcal{A}$ be an L'-structure that is elementarily equivalent to Γ. Then the connected components of $\mathcal{A}$ are precisely the maximal antichains of $\mathcal{A}$ with respect to $\lhd$.

$\because$ Two connected vertexes in Γ have path of length at most 4.

Then $\lhd$ induces an order into the connected components of $\mathcal{A}$ and it is denoted also by $\lhd$.

Review of Shelah's construction of models

Suppose that $\mathfrak{d} < \mathfrak{v}^{\forall}$. Then Γ and Γ_{NS} witness $\neg\,\mathsf{KT}(\aleph_0)$. In fact, for any ultrafilters p, q over ω, the following statements hold.

In $(\Gamma_{\mathrm{NS}})^{\omega}/q$, it holds that

there are cofinally many connected components C such that:

$$(\exists \langle u_i : i < \mathfrak{d}\rangle \text{ with each } u_i \in C \cap U)$$
$$(\forall v \in C \cap V)(\exists i < \mathfrak{d})(u_i \not E\ v).$$

In Γ^{ω}/p, it holds that for every $\kappa < \mathfrak{v}^{\forall}$,

for every connected component C in a final segment:

$$(\forall \langle u_i : i < \kappa\rangle \text{ with each } u_i \in C \cap U)$$
$$(\exists v \in C \cap V)(\forall i < \kappa)(u_i\ E\ v).$$

Puting $\kappa = \mathfrak{d}$ gives $\Gamma^{\omega}/p \not\preceq (\Gamma_{\mathrm{NS}})^{\omega}/q$.

Our Modification

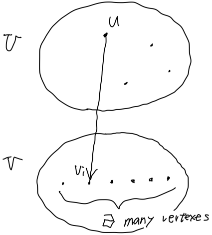
Suppose that $\mathfrak{d} < \mathfrak{c}^{\exists}$. Modify the definition of Γ by replacing $\langle \Delta_{n^2+1,n} : n \geq 1 \rangle$ with $\langle \Delta_{n^3,n} : n \geq 1 \rangle$. Then Γ and Γ_{NS} witness $\neg\,\mathrm{KT}(\aleph_0)$. In fact, for any ultrafilters p, q over ω, the following statements hold.

In $(\Gamma_{\mathrm{NS}})^{\omega}/q$, it holds that

there are cofinally many connected components C s.t.:

$(\exists \langle v_i : i < \mathfrak{d} \rangle$ with each $v_i \in C \cap V)$

$(\forall u \in C \cap U)(\exists i < \mathfrak{d})(u \; E \; v_i)$.



In Γ^{ω}/p, it holds that for every $\kappa < \mathfrak{c}^{\exists}$,

for every connected component C in a final segment:

$(\forall \langle v_i : i < \kappa \rangle$ with each $v_i \in C \cap V)$

$(\exists u \in C \cap U)(\forall i < \kappa)(u \not\!E \; v_i)$.

Puting $\kappa = \mathfrak{d}$ gives $\Gamma^{\omega}/p \not\cong (\Gamma_{\mathrm{NS}})^{\omega}/q$.

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SAT($\aleph_0$) implies $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ and $2^{<\mathfrak{c}} = \mathfrak{c}$

Review of the definiton of SAT

We say SAT(κ) **holds** if there exists an ultrafilter U over ω such that for every language $\mathcal{L}$ and every sequence of $\mathcal{L}$-structures $(\mathcal{A}_i)_{i\in\omega}$ with each $\mathcal{A}_i$ of size $\leq \kappa$, $\prod_{i\in\omega} \mathcal{A}_i/U$ is finite or $\mathfrak{c}$-saturated.

In the preprint, we also showed the converse: $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies SAT($\aleph_0$), But in this talk we don't deal with it.

SAT($\aleph_0$) implies $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$

We use the following lemma which characterizes $\mathrm{cov}(\mathcal{M})$.

Lemma (Bartoszyński)

$$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \iff (\forall X \subseteq \omega^\omega \text{ of size } < \mathfrak{c})(\exists S \in \prod_{i \in \omega} [\omega]^{\leq i}) \\ (\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$$

SAT($\aleph_0$) implies $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$

- Take an ultrafilter U that witnesses SAT($\aleph_0$).
- Fix $X \subseteq \omega^\omega$ of size $< \mathfrak{c}$.
- Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$ and define each $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\omega]^{\leq i}, \subseteq)$.
- For each $x \in \omega^\omega$, let $S_x = (\{x(i)\} : i \in \omega)$.
- In the ultraproduct $\mathcal{A}^* = \prod_{i \in \omega} \mathcal{A}_i / U$, consider a set of formulas with one free variable S defined by

$$p = \{[S_x] \subseteq S : x \in X\}.$$

SAT($\aleph_0$) implies $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$

- This p is finitely satisfiable and the number of parameters that occur in p is $< \mathfrak{c}$.
- In order to check finitely satisfiability, take finitely many reals $x_0, \dots, x_m$. Then a slalom S defined by $S(n) = \{x_0(n), \dots, x_m(n)\}$ for $n \geq m$ covers $x_0, \dots, x_m$.
- Therefore by SAT($\aleph_0$), we can take $[S] \in \mathcal{A}^*$ that realizes p.
- This S satisfies $(\forall x \in X)(\{n \in \omega : x(n) \in S(n)\} \in U)$, so $(\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$ $\square$

SAT($\aleph_0$) implies $2^{<\mathfrak{c}} = \mathfrak{c}$

Take an ultrafilter U over ω that witnesses SAT($\aleph_0$). Fix $\kappa < \mathfrak{c}$.
Put $\mathcal{L} = \{\subseteq\}$ and define an $\mathcal{L}$-structure $\mathcal{A}$ by $\mathcal{A} = ([\omega]^{<\omega}, \subseteq)$.
Put $\mathcal{A}^* = \mathcal{A}^\omega / U$.
Define a map $\iota \colon \omega^\omega / U \to \mathcal{A}^*$ by $\iota([x]) = [\langle \{x(n)\} : n \in \omega \rangle]$.
By SAT($\aleph_0$), $|\omega^\omega / U| = \mathfrak{c}$. Take a subset F of ω^ω / U of size κ.
For each $X \subseteq F$, let p_X be a set of formulas with a free variable z defined by

$$p_X = \{\iota(y) \subseteq z : y \in X\} \cup \{\iota(y) \not\subseteq z : y \in F \setminus X\}$$

Each p_X is finitely satisfiable.

SAT($\aleph_0$) implies $2^{<\mathfrak{c}} = \mathfrak{c}$

$$p_X = \{\iota(y) \subseteq z : y \in X\} \cup \{\iota(y) \not\subseteq z : y \in F \setminus X\}.$$

Claim: Each p_X is finitely satisfiable.

$\because$ Take $[x_0], \ldots, [x_n] \in X$ and $[y_0], \ldots, [y_m] \in F \setminus X$. Put $z(i) = \{x_0(i), \ldots, x_n(i)\}$. Then $\iota([x_0]), \ldots, \iota([x_n]) \subseteq_U [z]$. In order to prove $\iota([y_j]) \not\subseteq_U [z]$ for each $j \leq m$, suppose that $\{i \in \omega : y_j(i) \in z(i)\} \in U$. Then for each $i \in \omega$, there is a $k_i \leq n$ such that $\{i \in \omega : y_j(i) = x_{k_i}(i)\} \in U$. Then there is a $k \leq n$ such that $\{i \in \omega : y_j(i) = x_k(i)\} \in U$. This implies $[y_j] = [x_k]$. Contradiction! //

SAT($\aleph_0$) implies $2^{<\mathfrak{c}} = \mathfrak{c}$

By SAT($\aleph_0$), for each $X \subseteq F$, take $[z_X] \in \mathcal{A}^*$ that realizes p_X. For $X, Y \subseteq F$ with $X \neq Y$, we have $[z_X] \neq [z_Y]$. So $2^\kappa = |\{[z_X] : X \subseteq F\}| \leq |\mathcal{A}^*| = \mathfrak{c}$. Therefore we have proved $2^{<\mathfrak{c}} = \mathfrak{c}$. □

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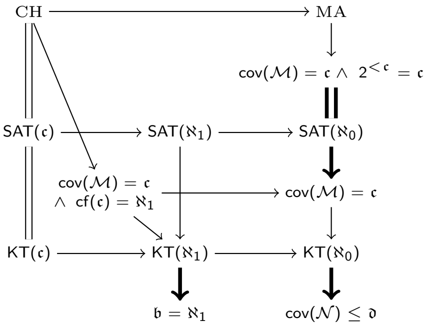
Open questions



1. Can CH and $\mathrm{SAT}(\aleph_1)$ be separated?
2. Does $\mathrm{KT}(\aleph_1)$ imply a stronger hypothesis than $\mathfrak{b} = \aleph_1$? Especially does $\mathrm{KT}(\aleph_1)$ imply $\mathrm{non}(\mathcal{M}) = \aleph_1$?
3. Does $\mathrm{KT}(\aleph_1)$ imply a hypothesis that some cardinal invariant is large?
4. Can $\mathrm{KT}(\aleph_0)$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ be separated?

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$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies SAT($\aleph_0$)

Note that $2^{<\mathfrak{c}} = \mathfrak{c}^{<\mathfrak{c}}$. So $2^{<\mathfrak{c}} = \mathfrak{c}$ implies $\mathfrak{c}$ is regular.
This proof is based on Elluntuck–Rucker.

Let $\langle b_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of ω^ω. Let $\langle (L_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < \mathfrak{c} \rangle$ be an enumeration of triples $(L, \mathcal{B}, \Delta)$ such that L is a countable language, $\mathcal{B} = \langle \mathcal{A}_i : i \in \omega \rangle$ is a sequence of L-structures with universe ω and Δ is a subset of $\mathrm{Fml}(L^+)$ with $|\Delta| < \mathfrak{c}$. Here $L^+ = L \cup \{c_\alpha : \alpha < \mathfrak{c}\}$ where the c_α's are new constant symbols and $\mathrm{Fml}(L^+)$ is the set of all L^+ formulas with one free variable. Here we used the assumption $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$. And ensure each $(L, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies $\mathrm{SAT}(\aleph_0)$

For $\mathcal{B}_\xi = \langle \mathcal{A}_i^\xi : i \in \omega \rangle$, put $\mathcal{B}_\xi(i) = (\mathcal{A}_i^\xi, b_0(i), b_1(i), \dots)$, which is a L^+-structure.

Let $\langle X_\xi : \xi < \mathfrak{c} \rangle$ be an enumeration of $\mathcal{P}(\omega)$.

We construct a sequence $\langle F_\xi : \xi < \mathfrak{c} \rangle$ of filters inductively so that the following properties hold:

1. F_0 is the filter consisting of all cofinite subsets of ω.
2. $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha<\xi} F_\alpha$ for ξ limit.
3. $X_\xi \in F_{\xi+1}$ or $\omega \setminus X_\xi \in F_{\xi+1}$.
4. F_ξ is generated by $< \mathfrak{c}$ members.
5. If

   $$\text{for all } \Gamma \subseteq \Delta_\xi \text{ finite, } \{i \in \omega : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi, \qquad (*)$$

   then there is a $f \in \omega^\omega$ such that for all $\varphi \in \Delta_\xi$, $\{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies $\mathrm{SAT}(\aleph_0)$

Suppose we have constructed F_ξ. We construct $F_{\xi+1}$. Let F'_ξ be a generating subset of F_ξ with $|F'_\xi| < \mathfrak{c}$. If $(*)$ is false, let $F_{\xi+1}$ be the filter generated by $F'_\xi \cup \{X_\xi\}$ or $F'_\xi \cup \{\omega \setminus X_\xi\}$. Suppose $(*)$. Put $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$. For $n \in \omega$, put

$$D_n = \{p \in \mathbb{P} : n \in \operatorname{dom} p\}.$$

For $A \in F'_\xi$ and $\varphi_1, \ldots, \varphi_n \in \Delta_\xi$, put

$$E_{A,\varphi_1,\ldots,\varphi_n} = \{p \in \mathbb{P} : \ (\exists k \in \operatorname{dom} p \cap A) \\ (p(k) \text{ satisfies } \varphi_1, \ldots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\}.$$

By $(*)$, each D_n and each $E_{A,\varphi_1,\ldots,\varphi_n}$ is dense. By using $\mathrm{MA}(\mathrm{Cohen})$, take a generic filter $G \subseteq \mathbb{P}$ with respect to above dense sets. Put $f = \bigcup G$. Then $F''_\xi := F'_\xi \cup \{Y_\varphi : \varphi \in \Delta_\xi\}$ satisfies finite intersection property, where $Y_\varphi = \{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\}$. Let $F_{\xi+1}$ be the filter generated by $F''_\xi \cup \{X_\xi\}$ or $F''_\xi \cup \{\omega \setminus X_\xi\}$.

$\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies SAT($\aleph_0$)

We have constructed $\langle F_\xi : \xi < \mathfrak{c} \rangle$. The resulting ultrafilter $F = \bigcup_{\xi<\mathfrak{c}} F_\xi$ witnesses SAT.

Our Modification

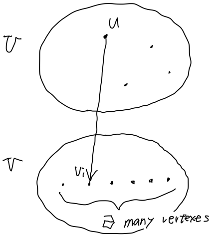
Suppose that $\mathfrak{d} < \mathfrak{c}^{\exists}$. Modify the definition of Γ by replacing $\langle \Delta_{n^2+1,n} : n \geq 1 \rangle$ with $\langle \Delta_{n^3,n} : n \geq 1 \rangle$. Then Γ and Γ_{NS} witness $\neg\,\mathrm{KT}(\aleph_0)$. In fact, for any ultrafilters p, q over ω, the following statements hold.

In $(\Gamma_{\mathrm{NS}})^{\omega}/q$, it holds that

there are cofinally many connected components C s.t.:

$(\exists \langle v_i : i < \mathfrak{d} \rangle$ with each $v_i \in C \cap V)$

$(\forall u \in C \cap U)(\exists i < \mathfrak{d})(u \mathrel{E} v_i)$.



In Γ^{ω}/p, it holds that for every $\kappa < \mathfrak{c}^{\exists}$,

for every connected component C in a final segment:

$(\forall \langle v_i : i < \kappa \rangle$ with each $v_i \in C \cap V)$

$(\exists u \in C \cap U)(\forall i < \kappa)(u \not\mathrel{E} v_i)$.

Puting $\kappa = \mathfrak{d}$ gives $\Gamma^{\omega}/p \not\simeq (\Gamma_{\mathrm{NS}})^{\omega}/q$.

Modified proof: $(\Gamma_{\mathrm{NS}})^\omega/q$ side

In $(\Gamma_{\mathrm{NS}})^\omega/q$, it holds that

there are cofinally many connected components C such that:
$(\exists\langle v_i : i < \mathfrak{d}\rangle$ with each $v_i \in C \cap V)$
$(\forall u \in C \cap U)(\exists i < \mathfrak{d})(u\ E\ v_i)$.

First, observe that every infinite connected component C of Γ_{NS} satisfies the following:

$(\forall F \subseteq C \cap U$ finite$)(\exists v \in C \cap V)($each point in F has an edge to $v)$.

Modified proof: $(\Gamma_{\mathrm{NS}})^\omega/q$ side

Claim

Let $\langle \Delta_n : n \in \omega \rangle$ be a sequence of bipartite graphs with $|U^{\Delta_n}| = |V^{\Delta_n}| = \aleph_0$. Suppose that for each $n \in \omega$,

$$(\forall F \subseteq U^{\Delta_n} \text{ finite})(\exists v \in V^{\Delta_n})(v \text{ has an edge to each point in } F).$$

Then for every ultraproduct $R := \prod_{n\in\omega} \Delta_n/q$, we have

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in V^R)(\forall u \in U^R)(\exists i < \mathfrak{d})(u\ E^R\ v_i).$$

$\because$ We may assume that each $U^{\Delta_n} = \omega$. Let $\{f_i : i < \mathfrak{d}\}$ be a cofinal subset of $(\omega^\omega, <^*)$. For each $n, m \in \omega$, take $v_{n,m} \in V^{\Delta_n}$ that is connected with first m points in U^{Δ_n}. For $i < \mathfrak{d}$, put

$$v_i = [\langle v_{n,f_i(n)} : n \in \omega \rangle].$$

Let $[u] \in U^R$. Consider u as an element of ω^ω. Take f_i that dominates u. Then we have

$$\{n \in \omega : u\ E^{\Delta_n}\ v_{n,f_i(n)}\} \in q.$$

Therefore $[u]\ E^R\ v_i$. //

Modified proof: $(\Gamma_{\mathrm{NS}})^\omega/q$ side

Claim (showed in the previous page)

Let $\langle \Delta_n : n \in \omega \rangle$ be a sequence of bipartite graphs with $|U^{\Delta_n}| = |V^{\Delta_n}| = \aleph_0$.
Suppose that for each $n \in \omega$,

$$(\forall F \subseteq U^{\Delta_n} \text{ finite})(\exists v \in V^{\Delta_n})(v \text{ has an edge to each point in } F).$$

Then for every ultraproduct $R := \prod_{n\in\omega} \Delta_n/q$, we have

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in V^R)(\forall u \in U^R)(\exists i < \mathfrak{d})(u \mathrel{E^R} v_i).$$

In $(\Gamma_{\mathrm{NS}})^\omega/q$, it holds that

there are cofinally many connected components C such that:

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in C \cap V)$$
$$(\forall u \in C \cap U)(\exists i < \mathfrak{d})(u \mathrel{E} v_i).$$

This statement follows from the first observation and Claim.

Modified proof: Γ^ω/p side

In Γ^ω/p, it holds that for every $\kappa < \mathfrak{c}^\exists$,

for every connected component C in a final segment:
$(\forall\langle v_i : i < \kappa\rangle$ with each $v_i \in C \cap V)$
$(\exists u \in C \cap U)(\forall i < \kappa)(u \not\trianglelefteq v_i)$.

Put $P = \Gamma^\omega/p$. Let $f\colon \omega \to \Gamma$ satisfy $f(n) \in G_n$ for all n. Let C_0 be the connected component that $[f]$ belongs to. Take a connected component C such that $C_0 \triangleleft C$ and an element $g \in C$. Take a function $h\colon \omega \to \omega$ such that $\{n \in \omega : g(n) \in G_{h(n)}\} \in q$. Then $A := \{n \in \omega : h(n) \geq n\} \in q$. Put $h'(n) = \max\{h(n), n\}$. Take $\langle [v_i] : i < \kappa\rangle$ with each $[v_i] \in C \cap V^P$. Then

$$B_i := \{n \in \omega : v_i(n) \in G_{h(n)} \cap V^\Gamma\} \in q.$$

Modified proof: Γ^ω/p side

In Γ^ω/p, it holds that for every $\kappa < \mathfrak{c}^\exists$,

for every connected component C in a final segment:
$(\forall \langle v_i : i < \kappa \rangle$ with each $v_i \in C \cap V)$
$(\exists u \in C \cap U)(\forall i < \kappa)(u \not\sqsubseteq v_i)$.

Take v_i' such that $v_i'(n) = v_i(n)$ for $n \in A_i$ and $v_i'(n) \in [h'(n)^3]^{\leq h'(n)}$ for $n \in \omega$. The assumption $\kappa < \mathfrak{c}^\exists$ and the calculation

$$\sum_{n \geq 1} \frac{h'(n)}{h'(n)^3} = \sum_{n \geq 1} \frac{1}{h'(n)^2} \leq \sum_{n \geq 1} \frac{1}{n^2} < \infty$$

give a $x \in \prod h'$ such that for all $i < \kappa$, $(\forall^\infty n)(x(n) \notin v_i'(n))$.
For each $i < \kappa$, take n_i such that $(\forall n \geq n_i)(x(n) \notin v_i'(n))$.

Modified proof: Γ^ω/p side

In Γ^ω/p, it holds that for every $\kappa < \mathfrak{c}^\exists$,

for every connected component C in a final segment:
$(\forall \langle v_i : i < \kappa\rangle$ with each $v_i \in C \cap V)$
$(\exists u \in C \cap U)(\forall i < \kappa)(u \not\!E\ v_i)$.

Take a point $[u] \in U^P$ such that $u(n) = x(n)$ for all $n \in A$.
Then for all $i < \kappa$ we have

$$\{n \in \omega : u(n) \not\!E^\Gamma\ v_i(n)\} \supseteq A \cap B_i \cap [n_i, \omega) \in q.$$

Therefore $[u] \not\!E^P\ [v_i]$ for all $i < \kappa$.

So we have that $\mathrm{KT}(\aleph_0)$ implies $\mathfrak{c}^\exists \leq \mathfrak{d}$. □