Goldstern's principle

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Goldstern's theorem

(total domination order) For $x, x' \in \omega^\omega$, define a relation $x \leq x'$ by $(\forall n \in \omega)(x(n) \leq x'(n))$.

In 1993, Martin Goldstern proved the following theorem.

Goldstern's theorem

Let $A \subseteq \omega^\omega \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has Lebesgue measure 0. Also, assume $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$, which is called monotonicity property. Then $\bigcup_{x \in \omega^\omega} A_x$ has also Lebesgue measure 0.

He used the Shoenfield absoluteness theorem and the random forcing to show this theorem. Also he applied this theorem to uniform distribution theory.

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The principle GP(Γ)

Definition

Let Γ be a pointclass. Then GP(Γ) means the following statement: Let $A \subseteq \omega^\omega \times 2^\omega$ be in Γ. Assume that for each $x \in \omega^\omega$, A_x has Lebesgue measure 0. Also suppose the monotonicity property. Then $\bigcup_{x \in \omega^\omega} A_x$ has also Lebesgue measure 0.

Goldstern's theorem says that GP($\boldsymbol{\Sigma}^1_1$) holds.

Note that if we replace the domination order $\leq$ by the almost domination order $\leq^*$, then the principle does not change.

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In choiceless context

The symbol “all” denotes the class of all subsets of Polish spaces. In choiceless context, we showed the following two theorems.

Theorem A (G.) (ZF)

AD implies GP(all).

Theorem B (G.) (ZF+CC)

If the measure uniformization holds then GP(all) holds. In particular, GP(all) holds in the Solovay model.

Proof of Theorem B

Theorem B (G.) (ZF+CC)

If the measure uniformization holds then GP(all) holds.

Definition The **measure uniformization** holds if for every family $\langle B_x : x \in \mathbb{R}\rangle$ with $\varnothing \neq B_x \subseteq \mathbb{R}$, there is a Lebesuge measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x : f(x) \notin B_x\}$ is null.

Proof of Theorem B. Let $\langle A_x : x \in \omega^\omega\rangle$ be monotone and each A_x is null. Suppose $\bigcup_{x\in\omega^\omega} A_x$ is not null. Then there is a compact positive set $K \subseteq \bigcup_{x\in\omega^\omega} A_x$. Use the measure uniformization for $\{(y, x) \in K \times \omega^\omega : y \in A_x\}$. Shrinking K and using Lusin's theorem, we can take continuous $f : K \to \omega^\omega$ such that $y \in A_{f(y)}$ for every $y \in K$. Since $f[K]$ is compact, we can take an upper bound x of $f[K]$. Then we have $K \subseteq A_x$, which is a contradiction. $\square$

Independence of GP(all) with ZFC

Is GP(all) consistent with the axiom of choice...?

$\rightsquigarrow$ Yes!

Theorem C (G.)

GP(all) is independent from ZFC.

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Theorem C (G.)

GP(all) is independent from ZFC.

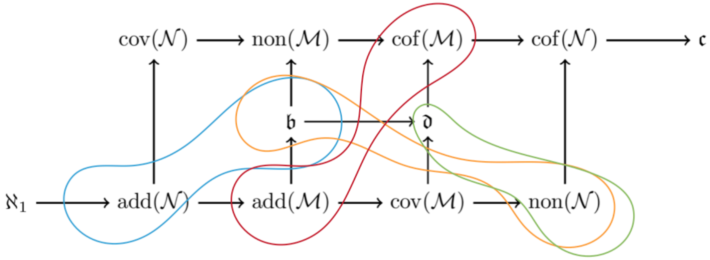
Consistency of $\neg\,$GP(all)

Theorem (G.)

Assume that at least one of the following four conditions holds:

$$\mathrm{add}(\mathcal{N}) = \mathfrak{b},\ \mathrm{non}(\mathcal{N}) = \mathfrak{b},\ \mathrm{non}(\mathcal{N}) = \mathfrak{d} \text{ or } \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}).$$

Then $\neg\,$GP(all) holds. In particular CH implies $\neg\,$GP(all).



Consistency of GP(all)

Theorem (G.)

If ZFC is consistent then so is $\mathrm{ZFC} + \mathsf{GP}(\mathrm{all})$.

In fact, "The Laver model" satisfies GP(all).

equal to $\aleph_2$ in the Laver model

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

equal to $\aleph_1$ in the Laver model

Theorem D (G.)

$\mathrm{GP}(\boldsymbol{\Pi}^1_1)$ holds.

Connection to regularity properties

Theorem E (G.)

$\mathbf{\Sigma}^1_2$ Lebesuge measurability implies $\mathrm{GP}(\mathbf{\Sigma}^1_2)$. Moreover, $\mathrm{GP}(\mathbf{\Delta}^1_2)$ implies that for every real a there is a dominating real over $L[a]$.

$$
\begin{tikzcd}
\mathbf{\Delta}^1_2(\mathbb{B}) \arrow[rrrrr] & & & & & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{S}) \\ \mathbf{\Delta}^1_2(\mathbb{S})\end{array} \\
& & & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{L}) \\ \mathbf{\Delta}^1_2(\mathbb{L})\end{array} \arrow[r] & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{M}) \\ \mathbf{\Delta}^1_2(\mathbb{M})\end{array} \arrow[ur] & \\
& & {\color{red}\mathrm{GP}(\mathbf{\Delta}^1_2)} \arrow[ur, red] & & & \\
& {\color{red}\mathrm{GP}(\mathbf{\Sigma}^1_2)} \arrow[ur, red] & & & & \\
\mathbf{\Sigma}^1_2(\mathbb{B}) \arrow[uuuu] \arrow[ur, red] \arrow[rrr] & & & \mathbf{\Sigma}^1_2(\mathbb{C}) \arrow[uuu] \arrow[r] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[uuu] &
\end{tikzcd}
$$

Connection to regularity properties

Theorem E (G.)

$\mathbf{\Sigma}^1_2$ Lebesuge measurability implies $\mathrm{GP}(\mathbf{\Sigma}^1_2)$. Moreover, $\mathrm{GP}(\mathbf{\Delta}^1_2)$ implies that for every real a there is a dominating real over $L[a]$.

$$
\begin{tikzcd}
\text{$\forall r \in \mathbb{R}$ [there is a random real over $L[r]$]} & \mathbf{\Delta}^1_2(\mathbb{B}) \arrow[rrrr, "\text{$\forall r \in \mathbb{R}$ [there is a dominating real over $L[r]$]}"] & & & & \begin{matrix}\mathbf{\Sigma}^1_2(\mathbb{S})\\ \mathbf{\Delta}^1_2(\mathbb{S})\end{matrix} \\
& & & \begin{matrix}\mathbf{\Sigma}^1_2(\mathbb{L})\\ \mathbf{\Delta}^1_2(\mathbb{L})\end{matrix} \arrow[r] & \begin{matrix}\mathbf{\Sigma}^1_2(\mathbb{M})\\ \mathbf{\Delta}^1_2(\mathbb{M})\end{matrix} \arrow[ur] & \text{$\forall r \in \mathbb{R}$ [there is an unbounded real over $L[r]$]} \\
& & \color{red}{\mathrm{GP}(\mathbf{\Delta}^1_2)} \arrow[ur, red] & & & \\
& & \color{red}{\mathrm{GP}(\mathbf{\Sigma}^1_2)} \arrow[u, red] & & & \\
\text{$\forall r \in \mathbb{R}$ [there are measure 1 many random reals over $L[r]$]} & \mathbf{\Sigma}^1_2(\mathbb{B}) \arrow[uuuu] \arrow[ur, red] \arrow[rr] & & \mathbf{\Sigma}^1_2(\mathbb{C}) \arrow[uuu] \arrow[r] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[uuu] & \text{$\forall r \in \mathbb{R}$ [there is a Cohen real over $L[r]$]} \\
& & & \text{$\forall r \in \mathbb{R}$ [there are comeager many Cohen reals over $L[r]$]} & &
\end{tikzcd}
$$

Connection to regularity properties

Theorem F (G.)

$\mathbf{\Delta}^1_2(\mathbb{B}) \wedge \mathbf{\Sigma}^1_2(\mathbb{L})$ implies $\mathrm{GP}(\mathbf{\Pi}^1_2)$.

$$
\begin{tikzcd}
\mathbf{\Sigma}^1_2(\mathbb{B}) \arrow[r] \arrow[dd] & \mathrm{GP}(\mathbf{\Sigma}^1_2) \arrow[dr] & & \\
& & \mathrm{GP}(\mathbf{\Delta}^1_2) \arrow[r] & \mathbf{\Sigma}^1_2(\mathbb{L}) \\
\mathbf{\Delta}^1_2(\mathbb{B}) \wedge \mathbf{\Sigma}^1_2(\mathbb{L}) \arrow[r] & \mathrm{GP}(\mathbf{\Pi}^1_2) \arrow[ur] & &
\end{tikzcd}
$$

Open questions

1. Is $\mathrm{ZFC} + (\mathfrak{c} \geq \aleph_3) + \mathsf{GP}(\mathrm{all})$ consistent?
2. For some $n \geq 2$ (or for every $n \geq 2$), can we separate $\mathsf{GP}(\mathbf{\Sigma}^1_{n+1})$ and $\mathsf{GP}(\mathbf{\Sigma}^1_n)$ without using large cardinals? Also can we separate $\mathsf{GP}(\mathbf{\Sigma}^1_n)$ and $\mathsf{GP}(\mathbf{\Pi}^1_n)$?
3. Can we separate each of the implications of the figure on the previous slide?

Models

	$\mathbf{\Sigma}^1_2(\mathbb{B})$	$\mathbf{\Delta}^1_2(\mathbb{B})$	$\mathbf{\Sigma}^1_2(\mathbb{L})$	GP(all)	GP($\mathbf{\Sigma}^1_2$)	GP($\mathbf{\Pi}^1_2$)	GP($\mathbf{\Delta}^1_2$)
Cohen	NO	NO	NO	NO	NO	NO	NO
random	NO	YES	NO	NO	NO	NO	NO
amoeba	YES	YES	YES	NO	YES	YES	YES
Laver	NO	NO	YES	YES	YES	YES	YES
Hechler	NO	NO	YES	NO	?	?	?
Mathias	NO	NO	YES	NO	?	?	?
Laver*random	NO	YES	YES	YES	YES	YES	YES
Hechler*random	NO	YES	YES	NO	?	YES	YES
Mathias*random	NO	YES	YES	NO	?	YES	YES
Ramsey filter guided Laver	NO	NO	YES	?	?	?	?

References

[Gol93] Martin Goldstern. “An Application of Shoenfield’s Absoluteness Theorem to the Theory of Uniform Distribution.”. In: *Monatshefte für Mathematik* 116.3-4 (1993), pp. 237–244.

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