Cardinal invariants and the Borel conjecture

Tatsuya Goto

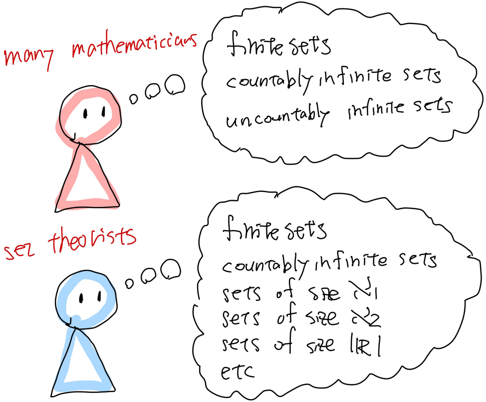
Kobe University

April 26, 2023
at Kobe Logic Seminar

1. Introduction to set theory of reals
2. The Borel conjecture
3. The problem the speaker wants to solve

Cardinalities

There is a difference in the precision of the idea of cardinalities in many mathematicians and set theorists.

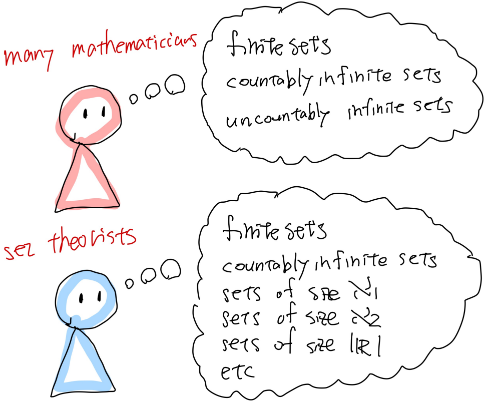


※ $\aleph_1$ is the successor cardinal of $\aleph_0 = |\omega|$, $\aleph_2$ is the successor cardinal of $\aleph_1$, and so on.

※ The cardinality of the continuum $|\mathbb{R}|$ is not determined by ZFC. It could be $\aleph_1$ or it could be $\aleph_2$. It could be much larger.

Cardinalities

There is a difference in the precision of the idea of cardinalities in many mathematicians and set theorists.

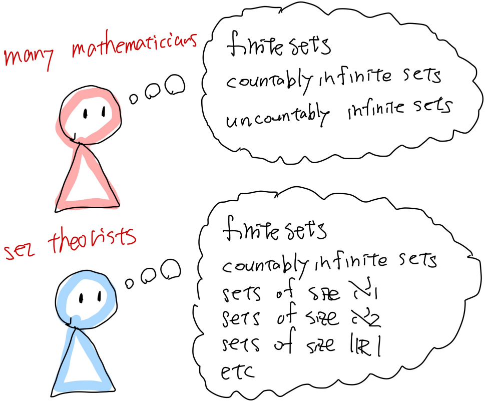


※ $\aleph_1$ is the successor cardinal of $\aleph_0 = |\omega|$, $\aleph_2$ is the successor cardinal of $\aleph_1$, and so on.

※ The cardinality of the continuum $|\mathbb{R}|$ is not determined by ZFC. It could be $\aleph_1$ or it could be $\aleph_2$. It could be much larger.

Cardinalities

There is a difference in the precision of the idea of cardinalities in many mathematicians and set theorists.



※ $\aleph_1$ is the successor cardinal of $\aleph_0 = |\omega|$, $\aleph_2$ is the successor cardinal of $\aleph_1$, and so on.

※ The cardinality of the continuum $|\mathbb{R}|$ is not determined by ZFC. It could be $\aleph_1$ or it could be $\aleph_2$. It could be much larger.

A motivational question

Question A

How many sets of measure 0 are needed to cover the real line?

Many mathematicians may answer that it is uncountably infinite and then they consider the question answered completely.

But set theorists consider this answer is not complete.

A motivational question

Question A

How many sets of measure 0 are needed to cover the real line?

Many mathematicians may answer that it is uncountably infinite and then they consider the question answered completely.

But set theorists consider this answer is not complete.

A motivational question

Question A

How many sets of measure 0 are needed to cover the real line?

Many mathematicians may answer that it is uncountably infinite and then they consider the question answered completely.

But set theorists consider this answer is not complete.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Why this answer is not complete?

This is because there are many uncountably infinite cardinals.

Moreover, Gödel and Cohen showed that the continuum hypothesis ($|\mathbb{R}| = \aleph_1$) is independent from ZFC.

Thus, set theorists was more deeply interested in issues such as:

- Is the answer of Question A $\aleph_1$?
- Is the answer of Question A $|\mathbb{R}|$?

As soon as Cohen invented the forcing method, set theorists showed that the answers to these questions are both ZFC independent.

Cardinal invariants

To better understand the answer to Question A, name it $\mathrm{cov}(\mathcal{N})$.
That is, let

$$\mathrm{cov}(\mathcal{N}) = \min\{\kappa \text{ cardinal} : \mathbb{R} \text{ can be covered by } \kappa \text{ many sets of measure } 0\}.$$

We name various definable cardinals using the structure of the real line similarly. They are called **cardinal invariants**.

Cardinal invariants

To better understand the answer to Question A, name it $\mathrm{cov}(\mathcal{N})$. That is, let

$$\mathrm{cov}(\mathcal{N}) = \min\{\kappa \text{ cardinal} : \mathbb{R} \text{ can be covered by } \kappa \text{ many sets of measure } 0\}.$$

We name various definable cardinals using the structure of the real line similarly. They are called **cardinal invariants**.

Cardinal invariants

To better understand the answer to Question A, name it cov($\mathcal{N}$). That is, let

$$\text{cov}(\mathcal{N}) = \min\{\kappa \text{ cardinal} : \mathbb{R} \text{ can be covered by } \kappa \text{ many sets of measure } 0\}.$$

We name various definable cardinals using the structure of the real line similarly. They are called **cardinal invariants**.

Set theory of reals

We investigate whether a greater-than-or-equal-to or less-than-or-equal-to relationship can be shown between cardinal invariants, or whether a consistency of greater-than or less-than relationship can be established.

Investigating aspects of the infinite world in this way is what is done in the field of **set theory of reals**.

Set theory of reals

We investigate whether a greater-than-or-equal-to or less-than-or-equal-to relationship can be shown between cardinal invariants, or whether a consistency of greater-than or less-than relationship can be established.

Investigating aspects of the infinite world in this way is what is done in the field of **set theory of reals**.

Meager sets

A set $X \subseteq \mathbb{R}$ is called **nowhere dense** if the interior of the closure of X is the empty set.

A set $X \subseteq \mathbb{R}$ is called **meager** if X is a union set of countably many nowhere dense sets.

Meager sets

A set $X \subseteq \mathbb{R}$ is called **nowhere dense** if the interior of the closure of X is the empty set.
A set $X \subseteq \mathbb{R}$ is called **meager** if X is a union set of countably many nowhere dense sets.

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\mathrm{add}(I) := \min\{\kappa : I$ is not closed under union of size $\kappa\}$
- $\mathrm{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\mathrm{cov}(I) := \min\{\kappa : \mathbb{R}$ can be covered by κ many members of $I\}$.
- $\mathrm{cof}(I) := \min\{|J| : J$ is a base of $I\}$.

$$\begin{array}{ccccccc} & & & \mathrm{non}(I) & & & \\ & & \nearrow & & \searrow & & \\ \aleph_1 \longrightarrow & \mathrm{add}(I) & & & & \mathrm{cof}(I) \longrightarrow & \mathfrak{c} \\ & & \searrow & & \nearrow & & \\ & & & \mathrm{cov}(I) & & & \end{array}$$

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\text{add}(I) := \min\{\kappa : I \text{ is not closed under union of size } \kappa\}$
- $\text{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\text{cov}(I) := \min\{\kappa : \mathbb{R} \text{ can be covered by } \kappa \text{ many members of } I\}$.
- $\text{cof}(I) := \min\{|J| : J \text{ is a base of } I\}$.

$$
\begin{array}{ccccccc}
 & & & \text{non}(I) & & & \\
 & & \nearrow & & \searrow & & \\
\aleph_1 \longrightarrow & \text{add}(I) & & & & \text{cof}(I) & \longrightarrow \mathfrak{c} \\
 & & \searrow & & \nearrow & & \\
 & & & \text{cov}(I) & & &
\end{array}
$$

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\mathrm{add}(I) := \min\{\kappa : I$ is not closed under union of size $\kappa\}$
- $\mathrm{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\mathrm{cov}(I) := \min\{\kappa : \mathbb{R}$ can be covered by κ many members of $I\}$.
- $\mathrm{cof}(I) := \min\{|J| : J$ is a base of $I\}$.

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\mathrm{add}(I) := \min\{\kappa : I$ is not closed under union of size $\kappa\}$
- $\mathrm{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\mathrm{cov}(I) := \min\{\kappa : \mathbb{R}$ can be covered by κ many members of $I\}$.
- $\mathrm{cof}(I) := \min\{|J| : J$ is a base of $I\}$.

$$
\begin{array}{ccccccc}
 & & & \mathrm{non}(I) & & & \\
 & & \nearrow & & \searrow & & \\
\aleph_1 \longrightarrow & \mathrm{add}(I) & & & & \mathrm{cof}(I) & \longrightarrow \mathfrak{c} \\
 & & \searrow & & \nearrow & & \\
 & & & \mathrm{cov}(I) & & &
\end{array}
$$

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\mathrm{add}(I) := \min\{\kappa : I$ is not closed under union of size $\kappa\}$
- $\mathrm{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\mathrm{cov}(I) := \min\{\kappa : \mathbb{R}$ can be covered by κ many members of $I\}$.
- $\mathrm{cof}(I) := \min\{|J| : J$ is a base of $I\}$.

$$
\begin{array}{ccccccc}
 & & & \mathrm{non}(I) & & & \\
 & & \nearrow & & \searrow & & \\
\aleph_1 \longrightarrow & \mathrm{add}(I) & & & & \mathrm{cof}(I) & \longrightarrow \mathfrak{c} \\
 & & \searrow & & \nearrow & & \\
 & & & \mathrm{cov}(I) & & &
\end{array}
$$

Definition of some cardinal invariants (1)

$\mathcal{N}$ and $\mathcal{M}$ denotes the collections of Lebesgue measure 0 sets and meager sets respectively. Let I be any of $\mathcal{N}$ or $\mathcal{M}$.

- $\mathrm{add}(I) := \min\{\kappa : I \text{ is not closed under union of size } \kappa\}$
- $\mathrm{non}(I) := \min\{|A| : A \subseteq \mathbb{R}, A \notin I\}$.
- $\mathrm{cov}(I) := \min\{\kappa : \mathbb{R} \text{ can be covered by } \kappa \text{ many members of } I\}$.
- $\mathrm{cof}(I) := \min\{|J| : J \text{ is a base of } I\}$.

$$\begin{array}{ccccccccc} & & & & \mathrm{non}(I) & & & & \\ & & & \nearrow & & \searrow & & & \\ \aleph_1 & \longrightarrow & \mathrm{add}(I) & & & & \mathrm{cof}(I) & \longrightarrow & \mathfrak{c} \\ & & & \searrow & & \nearrow & & & \\ & & & & \mathrm{cov}(I) & & & & \end{array}$$

Definition of some cardinal invariants (2)

Let ω^ω denotes the set of all functions from ω to ω. Define a partial preorder $\leq^*$ into ω^ω by:

$$x \leq^* y \iff \text{for all but finitely many } n,\ x(n) \leq y(n).$$

$$(y \text{ dominates } x).$$

Definition of some cardinal invariants (2)

Let ω^ω denotes the set of all functions from ω to ω. Define a partial preorder $\leq^*$ into ω^ω by:

$$x \leq^* y \iff \text{for all but finitely many } n,\ x(n) \leq y(n).$$

$$(y \text{ dominates } x).$$

Definition of some cardinal invariants (3)

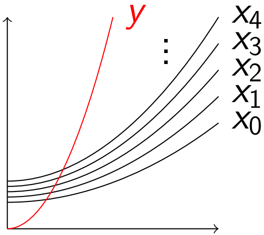
- $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ and there is no single $y \in \omega^\omega$ such that every $x \in F$ is dominated by $y\}$.
- $\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega$ and for all $y \in \omega^\omega$ there is $x \in F$ such that x dominates $y\}$.

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c}$$

Definition of some cardinal invariants (3)

- $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ and there is no single $y \in \omega^\omega$ such that every $x \in F$ is dominated by $y\}$.
- $\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega$ and for all $y \in \omega^\omega$ there is $x \in F$ such that x dominates $y\}$.

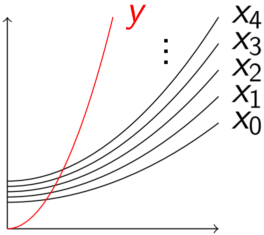
$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c}$$



Definition of some cardinal invariants (3)

- $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ and there is no single $y \in \omega^\omega$ such that every $x \in F$ is dominated by $y\}$.
- $\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega$ and for all $y \in \omega^\omega$ there is $x \in F$ such that x dominates $y\}$.

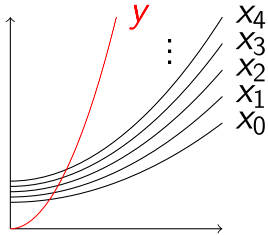
$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c}$$



Definition of some cardinal invariants (3)

- $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ and there is no single $y \in \omega^\omega$ such that every $x \in F$ is dominated by $y\}$.
- $\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega$ and for all $y \in \omega^\omega$ there is $x \in F$ such that x dominates $y\}$.

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c}$$



Cichoń's diagram

In the following diagram, the arrow drawn from a cardinal A to another cardinal B indicates that $A \leq B$ is provable from ZFC.

$$
\begin{array}{ccccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & \mathfrak{c} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}
$$

This diagram is complete in the sense that we can draw no more lines.

1. Introduction to set theory of reals

2. The Borel conjecture

3. The problem the speaker wants to solve

The Borel conjecture

Definition (strongly measure zero)

A set $A \subseteq \mathbb{R}$ is called a **strongly measure zero** set if for every sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive real numbers there is a sequence $\langle I_n : n \in \omega \rangle$ of intervals such that the length of I_n is smaller than ε_n for every n and $A \subseteq \bigcup_{n\in\omega} I_n$.

It holds that countable $\subseteq \mathcal{SN} \subseteq \mathcal{N}$.

The Borel conjecture

The **Borel conjecture** states that every strongly measure zero set is countable.

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. $\square$

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. $\square$

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. □

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. □

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. $\square$

The Galvin-Mycielski-Solovay theorem

The Galvin-Mycielski-Solovay theorem

$X \subseteq \mathbb{R}$ is strongly measure zero iff, for every $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

($\because$) We only show the easy direction $\Leftarrow$. Fix $\langle \varepsilon_n : n \in \omega \rangle$. Let $\mathbb{Q} = \{q_n : n \in \omega\}$. Put $I_n = (q_n - \varepsilon_n/2, q_n + \varepsilon_n/2)$. Then $M := \mathbb{R} \setminus \bigcup_n I_n$ is meager. So by the assumption, we can take $z \in \mathbb{R}$ such that $z \notin X + M$. This implies $X \subseteq (z + (-M))^{\mathrm{c}}$. Put $J_n = z + (-I_n)$. Then $X \subseteq \bigcup_n J_n$. $\square$

The Galvin-Mycielski-Solovay theorem

Corollary

Every set of reals of size $< \mathrm{cov}(\mathcal{M})$ is strongly measure zero.

($\because$) Let $X \subseteq \mathbb{R}, |X| < \mathrm{cov}(\mathcal{M})$. Let $M \in \mathcal{M}$. Then $\bigcup_{x \in X}(x + M) \neq \mathbb{R}$ by $x + M \in \mathcal{M}$ and $|X| < \mathrm{cov}(\mathcal{M})$. But $\bigcup_{x \in X}(x + M) = X + M$. □

We have that the Borel conjecture implies $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

The Galvin-Mycielski-Solovay theorem

Corollary

Every set of reals of size $< \mathrm{cov}(\mathcal{M})$ is strongly measure zero.

(∵) Let $X \subseteq \mathbb{R}, |X| < \mathrm{cov}(\mathcal{M})$. Let $M \in \mathcal{M}$. Then $\bigcup_{x \in X}(x + M) \neq \mathbb{R}$ by $x + M \in \mathcal{M}$ and $|X| < \mathrm{cov}(\mathcal{M})$. But $\bigcup_{x \in X}(x + M) = X + M$. □

We have that the Borel conjecture implies $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

The Galvin-Mycielski-Solovay theorem

Corollary

Every set of reals of size $< \mathrm{cov}(\mathcal{M})$ is strongly measure zero.

($\because$) Let $X \subseteq \mathbb{R}, |X| < \mathrm{cov}(\mathcal{M})$. Let $M \in \mathcal{M}$. Then $\bigcup_{x \in X}(x + M) \neq \mathbb{R}$ by $x + M \in \mathcal{M}$ and $|X| < \mathrm{cov}(\mathcal{M})$. But $\bigcup_{x \in X}(x + M) = X + M$. □

We have that the Borel conjecture implies $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

The Galvin-Mycielski-Solovay theorem

Corollary

Every set of reals of size $< \mathrm{cov}(\mathcal{M})$ is strongly measure zero.

($\because$) Let $X \subseteq \mathbb{R}, |X| < \mathrm{cov}(\mathcal{M})$. Let $M \in \mathcal{M}$. Then $\bigcup_{x \in X}(x + M) \neq \mathbb{R}$ by $x + M \in \mathcal{M}$ and $|X| < \mathrm{cov}(\mathcal{M})$. But $\bigcup_{x \in X}(x + M) = X + M$. $\square$

We have that the Borel conjecture implies $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

$\mathfrak{b} > \aleph_1$ is necessary

Fact

Assume $\mathfrak{b} = \aleph_1$. Then there is an uncoutanble strongly measure zero set.

We use the Cantor space 2^ω instead of $\mathbb{R}$. By $\mathfrak{b} = \aleph_1$, we can take an increasing unbounded family $F = \{f_\alpha : \alpha < \aleph_1\}$ of elements in ω^ω. Let $\iota : \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ be the homeomorphism defined by:

$$\iota(f) = 0^{(f(0))}{}^\frown 1^\frown 0^{(f(1))}{}^\frown 1^\frown \dots .$$

We claim that $\iota[F]$ is strongly measure zero.

$\mathfrak{b} > \aleph_1$ is necessary

Fact

Assume $\mathfrak{b} = \aleph_1$. Then there is an uncoutanble strongly measure zero set.

We use the Cantor space 2^ω instead of $\mathbb{R}$. By $\mathfrak{b} = \aleph_1$, we can take an increasing unbounded family $F = \{f_\alpha : \alpha < \aleph_1\}$ of elements in ω^ω. Let $\iota\colon \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ be the homeomorphism defined by:

$$\iota(f) = 0^{(f(0))} {}^\frown 1 {}^\frown 0^{(f(1))} {}^\frown 1 {}^\frown \ldots.$$

We claim that $\iota[F]$ is strongly measure zero.

$\mathfrak{b} > \aleph_1$ is necessary

Fact

Assume $\mathfrak{b} = \aleph_1$. Then there is an uncoutanble strongly measure zero set.

We use the Cantor space 2^ω instead of $\mathbb{R}$. By $\mathfrak{b} = \aleph_1$, we can take an increasing unbounded family $F = \{f_\alpha : \alpha < \aleph_1\}$ of elements in ω^ω. Let $\iota \colon \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ be the homeomorphism defined by:

$$\iota(f) = 0^{(f(0))} {}^\frown 1 {}^\frown 0^{(f(1))} {}^\frown 1 {}^\frown \dots.$$

We claim that $\iota[F]$ is strongly measure zero.

$\mathfrak{b} > \aleph_1$ is necessary

Fact

Assume $\mathfrak{b} = \aleph_1$. Then there is an uncoutanble strongly measure zero set.

We use the Cantor space 2^ω instead of $\mathbb{R}$. By $\mathfrak{b} = \aleph_1$, we can take an increasing unbounded family $F = \{f_\alpha : \alpha < \aleph_1\}$ of elements in ω^ω. Let $\iota \colon \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ be the homeomorphism defined by:

$$\iota(f) = 0^{(f(0))\frown}1^\frown 0^{(f(1))\frown}1^\frown \dots.$$

We claim that $\iota[F]$ is strongly measure zero.

$\mathfrak{b} > \aleph_1$ is necessary

Fact

Assume $\mathfrak{b} = \aleph_1$. Then there is an uncoutanble strongly measure zero set.

We use the Cantor space 2^ω instead of $\mathbb{R}$. By $\mathfrak{b} = \aleph_1$, we can take an increasing unbounded family $F = \{f_\alpha : \alpha < \aleph_1\}$ of elements in ω^ω. Let $\iota \colon \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ be the homeomorphism defined by:

$$\iota(f) = 0^{(f(0))\frown} 1^\frown 0^{(f(1))\frown} 1^\frown \ldots .$$

We claim that $\iota[F]$ is strongly measure zero.

$\mathfrak{b} > \aleph_1$ is necessary

Claim: $\iota[F]$ is strongly measure zero.

Fix a sequence $\langle n_k : k \in \omega \rangle$ of natural numbers. We have to find a sequence $\langle s_k : k \in \omega \rangle$ of elements in $2^{<\omega}$ with $|s_k| = n_k$ such that $\iota[F] \subseteq \bigcup_k [s_k]$.
Let $\mathbb{Q} = \{q_k : k \in \omega\}$ and let $s_{2k} = q_k \restriction n_{2k}$ (for each $k \in \omega$) and put $U = \bigcup_k [s_{2k}]$. It is enough to prove that $\iota[F] \setminus U$ is countable. So we shall show $F \cap Y$ is countable, where $Y = \iota^{-1}(2^\omega \setminus U)$.

$\mathfrak{b} > \aleph_1$ is necessary

Claim: $\iota[F]$ is strongly measure zero.

Fix a sequence $\langle n_k : k \in \omega \rangle$ of natural numbers. We have to find a sequence $\langle s_k : k \in \omega \rangle$ of elements in $2^{<\omega}$ with $|s_k| = n_k$ such that $\iota[F] \subseteq \bigcup_k [s_k]$.

Let $\mathbb{Q} = \{q_k : k \in \omega\}$ and let $s_{2k} = q_k \restriction n_{2k}$ (for each $k \in \omega$) and put $U = \bigcup_k [s_{2k}]$. It is enough to prove that $\iota[F] \setminus U$ is countable. So we shall show $F \cap Y$ is countable, where $Y = \iota^{-1}(2^\omega \setminus U)$.

$\mathfrak{b} > \aleph_1$ is necessary

Claim: $\iota[F]$ is strongly measure zero.

Fix a sequence $\langle n_k : k \in \omega \rangle$ of natural numbers. We have to find a sequence $\langle s_k : k \in \omega \rangle$ of elements in $2^{<\omega}$ with $|s_k| = n_k$ such that $\iota[F] \subseteq \bigcup_k [s_k]$.
Let $\mathbb{Q} = \{q_k : k \in \omega\}$ and let $s_{2k} = q_k \restriction n_{2k}$ (for each $k \in \omega$) and put $U = \bigcup_k [s_{2k}]$. It is enough to prove that $\iota[F] \setminus U$ is countable. So we shall show $F \cap Y$ is countable, where $Y = \iota^{-1}(2^\omega \setminus U)$.

$\mathfrak{b} > \aleph_1$ is necessary

Claim: $\iota[F]$ is strongly measure zero.

Fix a sequence $\langle n_k : k \in \omega \rangle$ of natural numbers. We have to find a sequence $\langle s_k : k \in \omega \rangle$ of elements in $2^{<\omega}$ with $|s_k| = n_k$ such that $\iota[F] \subseteq \bigcup_k [s_k]$.
Let $\mathbb{Q} = \{q_k : k \in \omega\}$ and let $s_{2k} = q_k \restriction n_{2k}$ (for each $k \in \omega$) and put $U = \bigcup_k [s_{2k}]$. It is enough to prove that $\iota[F] \smallsetminus U$ is countable.
So we shall show $F \cap Y$ is countable, where $Y = \iota^{-1}(2^\omega \smallsetminus U)$.

$\mathfrak{b} > \aleph_1$ is necessary

Claim: $\iota[F]$ is strongly measure zero.

Fix a sequence $\langle n_k : k \in \omega \rangle$ of natural numbers. We have to find a sequence $\langle s_k : k \in \omega \rangle$ of elements in $2^{<\omega}$ with $|s_k| = n_k$ such that $\iota[F] \subseteq \bigcup_k [s_k]$.
Let $\mathbb{Q} = \{q_k : k \in \omega\}$ and let $s_{2k} = q_k \restriction n_{2k}$ (for each $k \in \omega$) and put $U = \bigcup_k [s_{2k}]$. It is enough to prove that $\iota[F] \setminus U$ is countable. So we shall show $F \cap Y$ is countable, where $Y = \iota^{-1}(2^\omega \setminus U)$.

$\mathfrak{b} > \aleph_1$ is necessary

Setting: $F \subseteq \omega^\omega$ is an increasing unbounded family of size $\aleph_1$.
$\langle n_k \rangle_k$ is given, $s_{2k} = q_k \restriction n_{2k}$, $U = \bigcup_k [s_{2k}]$
$Y = \iota^{-1}(2^\omega \setminus U)$

Goal: $F \cap Y$ is countable.

Since $2^\omega \setminus U$ is a compact set and ι^{-1} is continuous, Y is also compact. So there is a g such that $y \leq^* g$ for every $y \in Y$. Because F is increasing and unbounded, $F \cap g{\downarrow}$ is countable. So $F \cap Y$ is countable. □

$\mathfrak{b} > \aleph_1$ is necessary

Setting: $F \subseteq \omega^\omega$ is an increasing unbounded family of size $\aleph_1$.
$\langle n_k \rangle_k$ is given, $s_{2k} = q_k \upharpoonright n_{2k}$, $U = \bigcup_k [s_{2k}]$
$Y = \iota^{-1}(2^\omega \setminus U)$
Goal: $F \cap Y$ is countable.

Since $2^\omega \setminus U$ is a compact set and ι^{-1} is continuous, Y is also compact. So there is a g such that $y \leq^* g$ for every $y \in Y$.
Because F is increasing and unbounded, $F \cap g{\downarrow}$ is countable. So $F \cap Y$ is countable. $\square$

$\mathfrak{b} > \aleph_1$ is necessary

Setting: $F \subseteq \omega^\omega$ is an increasing unbounded family of size $\aleph_1$.
$\langle n_k \rangle_k$ is given, $s_{2k} = q_k \restriction n_{2k}$, $U = \bigcup_k [s_{2k}]$
$Y = \iota^{-1}(2^\omega \smallsetminus U)$

Goal: $F \cap Y$ is countable.

Since $2^\omega \smallsetminus U$ is a compact set and ι^{-1} is continuous, Y is also compact. So there is a g such that $y \leq^* g$ for every $y \in Y$. Because F is increasing and unbounded, $F \cap g\downarrow$ is countable. So $F \cap Y$ is countable. □

$\mathfrak{b} > \aleph_1$ is necessary

Setting: $F \subseteq \omega^\omega$ is an increasing unbounded family of size $\aleph_1$.
$\langle n_k \rangle_k$ is given, $s_{2k} = q_k \restriction n_{2k}$, $U = \bigcup_k [s_{2k}]$
$Y = \iota^{-1}(2^\omega \setminus U)$
Goal: $F \cap Y$ is countable.

Since $2^\omega \setminus U$ is a compact set and ι^{-1} is continuous, Y is also compact. So there is a g such that $y \leq^* g$ for every $y \in Y$. Because F is increasing and unbounded, $F \cap g\downarrow$ is countable. So $F \cap Y$ is countable. □

Necessary conditions for the Borel conjecture

Fact

Each of the statement $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{b} > \aleph_1$ is a necessary condition for the Borel conjecture.

Although the invariants $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{b}$ were not defined at the time Laver published his paper, the speaker believes that Laver must have been making essentially the same observation.

This observation led Laver to define the Laver forcing to get the consistency of the Borel conjecture.

Necessary conditions for the Borel conjecture

Fact

Each of the statement $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{b} > \aleph_1$ is a necessary condition for the Borel conjecture.

Although the invariants $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{b}$ were not defined at the time Laver published his paper, the speaker believes that Laver must have been making essentially the same observation.

This observation led Laver to define the Laver forcing to get the consistency of the Borel conjecture.

Necessary conditions for the Borel conjecture

Fact

Each of the statement $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{b} > \aleph_1$ is a necessary condition for the Borel conjecture.

Although the invariants $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{b}$ were not defined at the time Laver published his paper, the speaker believes that Laver must have been making essentially the same observation.

This observation led Laver to define the Laver forcing to get the consistency of the Borel conjecture.

Laver's theorem

Laver's theorem

If ZFC is consistent, then so is $\mathrm{ZFC} +$ (the Borel conjecture).

Laver invented the Laver forcing to prove this theorem.

1. Introduction to set theory of reals

2. The Borel conjecture

3. The problem the speaker wants to solve

Some collections of small sets of reals

Let $\mathcal{SN}$ be the set of strong measure zero sets. Let $\mathcal{SM}$ be the set of **strongly meager sets**, that is

$$\mathcal{SM} = \{X \subseteq \mathbb{R} : \text{for every } N \in \mathcal{N},\ X + N \neq \mathbb{R}\}.$$

Let $I, J \subseteq \mathcal{P}(\mathbb{R})$. Define $(I, J)^* \subseteq \mathcal{P}(\mathbb{R})$ by

$$(I, J)^* = \{X \subseteq \mathbb{R} : \text{for every } A \in I,\ A + X \in J\}.$$

For $I \subseteq \mathcal{P}(\mathbb{R})$, define I^* by $I^* = (I, I)^*$.

$\mathcal{E}$

Let

$\mathcal{E} = \{X \subseteq \mathbb{R}$: X is covered by countably many closed measure 0 sets$\}$.

It holds that $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$.

The dual Borel conjecture

The dual Borel conjecture states that every strongly meager set is countable.

Carlson's theorem

If ZFC is consistent, then so is ZFC + (the dual Borel conjecture).

Carlson used the Cohen forcing to show this.

The problem the speaker wants to solve

Fact

$$
\begin{array}{ccccccc}
 & & & & \mathcal{SM} & \subsetneq & (\mathcal{E},\mathcal{M})^* \\
 & & \rotatebox{45}{$\subsetneq$} & & & & \rotatebox{-45}{$\subsetneq$} \\
\mathcal{N}^* = (\mathcal{M}\cap\mathcal{N})^* & \subsetneq & \mathcal{E}^* = \mathcal{M}^* & \subsetneq & (\mathcal{E},\mathcal{M}\cap\mathcal{N})^* & & \\
 & & & & & \rotatebox{45}{$\subsetneq$} & \\
 & & & & & & (\mathcal{E},\mathcal{N})^* = \mathcal{SN}
\end{array}.
$$

Problem

Is it consistent that $(\mathcal{E},\mathcal{M})^* =$ countable?

The problem the speaker wants to solve

Fact

Each of the statement $\mathrm{cov}(\mathcal{N}) = \aleph_1$, $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{b} > \aleph_1$ is a necessary condition for $(\mathcal{E}, \mathcal{M})^* =$ countable.

The claim about $\mathfrak{b}$ is due to Bartoszynski.

Approaches to the problem

It is consistent that both the Borel conjecture and the dual Borel conjecture hold simultaneously. So if we have

$$(\mathcal{E}, \mathcal{M})^* \subseteq \{X + Y : X \in \mathcal{SN}, Y \in \mathcal{SM}\},$$

then the problem is solved.

Another possible approach would be to read the proof of BC+dBC consistency and imitate that method.

Approaches to the problem

It is consistent that both the Borel conjecture and the dual Borel conjecture hold simultaneously. So if we have

$$(\mathcal{E},\mathcal{M})^* \subseteq \{X+Y : X \in \mathcal{SN}, Y \in \mathcal{SM}\},$$

then the problem is solved.

Another possible approach would be to read the proof of BC+dBC consistency and imitate that method.

References

[Bar03] Tomek Bartoszynski. "Remarks on small sets of reals". In: *Proceedings of the American Mathematical Society* 131.2 (2003), pp. 625–630.

[BJ95] Tomek Bartoszynski and Haim Judah. *Set Theory: on the structure of the real line*. CRC Press, 1995.

[Wei20] Tomasz Weiss. "On the algebraic union of strongly measure zero sets and their relatives with sets of real numbers". In: *Centenary of the Borel Conjecture* 755 (2020), pp. 115–133.

[Woh20] Wolfgang Wohofsky. "Borel Conjecture and dual Borel Conjecture (and other variants of the Borel Conjecture)". In: *Centenary of the Borel Conjecture* 755 (2020), pp. 135–227.