New preservation properties for the left part of Cichoń's diagram

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Conference on the occasion of Jörg Brendle's 60th birthday

joint work with Diego Mejía

First of all

Happy birthday, Jörg!

Me

Nagoya University (MSc; supervised by Yoshinobu)
→ Kobe University (Ph.D.; supervised by Brendle)
→ TU Wien (postdoc; fellowship by JSPS)

Introduction

Let's move on to math!

Motivation

In the context of separating cardinal invariants, it's easy to increase the target invariant in many cases. But it's often difficult to preserve other invariants. Preservation theorems help this task.

Two preservation theorems: setting

Let $\langle \sqsubseteq_n : n \in \omega \rangle$ be an increasing sequence of binary relations on ω^ω and let $\sqsubseteq = \bigcup_n \sqsubseteq_n$. Assume the following:

1. For each $n \in \omega$ and $y \in \omega^\omega$, the set $\{x \in \omega^\omega : x \sqsubseteq_n y\}$ is a closed set.
2. $\mathrm{dom}(\sqsubseteq)$ is a closed subset of ω^ω.
3. $\mathfrak{b}(\sqsubseteq) \geqslant \aleph_1$.

Two preservation theorems: statements

First Preservation Theorem (Shelah)

Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of proper forcing notions such that $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ preserves $\sqsubseteq$" (we define this notion later). Then P_δ also preserves $\sqsubseteq$. In particular $P_\delta \Vdash \mathfrak{d}(\sqsubseteq) = \aleph_1$ if $V \models \mathrm{CH}$.

Second Preservation Theorem (Judah–Repický)

Additionally, assume a mild assumption on $\langle \sqsubseteq_n : n \in \omega \rangle$. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ (δ is a limit ordinal) be a countable support iteration of proper forcing notions. If for each $\alpha < \delta$, P_α does not add a $\sqsubseteq$-dominating real, then P_δ neither does. In particular $P_\delta \Vdash \mathfrak{b}(\sqsubseteq) = \aleph_1$.

Two preservation theorems: applications

It is customary to preserve an invariant from the right side of Cichoń's diagram small using the First Preservation Theorem and to preserve an invariant from the left side of Cichoń's diagram small using the Second Preservation Theorem.

But sometimes the Second Preservation Theorem is inconvenient because this theorem does not help at successor steps.

Therefore, we consider relations $\sqsubseteq$ to preserve an invariant from the **left** side of Cichoń's diagram small using the **First** Preservation Theorem.

First preservation theorem

almost preserving

Definition (almost preserving)

A forcing notion P **almost preserves** $\sqsubseteq$ if whenever $N \prec H_\theta$ is a countable model such that $P, \sqsubseteq \in N$ and if y is a $\sqsubseteq$-dominating real over N, $p \in P \cap N$, then there is an N-generic condition $q \leqslant p$ forcing that y is a $\sqsubseteq$-dominating real over $N[\dot{G}]$.

Note that if P almost preserves $\sqsubseteq$, then P is proper and forces $\forall f \in \mathrm{dom}(\sqsubseteq) \cap V[\dot{G}]\ \exists g \in V\ f \sqsubseteq g$.

Preserving

Definition (preserving)

A forcing notion P **preserves** $\sqsubseteq$ if whenever $N \prec H_\theta$ is a countable model such that $P, \sqsubseteq \in N$ and if y is a $\sqsubseteq$-dominating real over N and $\langle p_n : n \in \omega \rangle \in N$ is a decreasing sequence of conditions interpreting $\langle \dot{f}_0, \ldots, \dot{f}_k \rangle \in N$ as $\langle f_0^*, \ldots, f_k^* \rangle$, then there is an N-generic condition $q \leqslant p_0$ forcing that y is a $\sqsubseteq$-dominating real over $N[\dot{G}]$ and $\forall n \in \omega \ \forall i \leqslant k \ (f_i^* \sqsubseteq_n y \rightarrow \dot{f}_i \sqsubseteq_n y)$.

Note that if P preserves $\sqsubseteq$, then P almost preserves $\sqsubseteq$.

Preserving

Lemma

Suppose $\{f : f \sqsubseteq_n g\}$ is relatively open in $\mathrm{dom}(\sqsubseteq)$ for every $n \in \omega$ and $g \in \omega^\omega$. If P almost preserves $\sqsubseteq$, then P preserves $\sqsubseteq$.

In particular, that $\sqsubseteq_n$ is clopen for every n implies the conclusion of this lemma. Fortunately, all our examples are such relations.

The picture of implications

preserves $\sqsubseteq$

$\downarrow$ (and back up, when $\sqsubseteq_n$ are clopen)

almost preserves $\sqsubseteq$

$\downarrow$

P does not add an unbounded real wrt $\sqsubseteq$

Preserving property is preserved by iteration

First Preservation Theorem (Goldstern, Shelah)

Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of proper forcing notions such that $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ preserves $\sqsubseteq$". Then P_δ also preserves $\sqsubseteq$.

Therefore, if $V \models \mathrm{CH}$ and $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration of proper forcing notions such that $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ preserves $\sqsubseteq$", then
$P_{\omega_2} \Vdash \mathfrak{d}(\sqsubseteq) = \aleph_1$ while $P_{\omega_2} \Vdash \mathfrak{c} = \aleph_2$.
Of course, when $\sqsubseteq_n$ $(n \in \omega)$ are clopen, then it suffices to check
$P_\alpha \Vdash$ "$\dot{Q}_\alpha$ almost preserves $\sqsubseteq$".

Tree relational system

Tree relational system: the definition

We say $R = (\lim T, Y, \sqsubseteq)$ is a **tree relational system** if the following conditions hold:

1. T is a countable well-pruned tree of sequences, height ω.
2. Y is an analytic set in some Polish space.
3. For $k \in \omega$, $\sqsubseteq_k^- \subseteq \lim T \times Y$ is an analytic set such that $\{x \in \lim T : x \sqsubseteq_k^- y\}$ is clopen for every $y \in Y$.

For $n \in \omega$, we define $x \sqsubseteq_n y$ if $x \sqsubseteq_k^-$ for some $k \leqslant n$.

4. $x \sqsubseteq y$ iff $x \sqsubseteq_n y$ for some n. This is equivalent to $x \sqsubseteq_k^- y$ for some k.
5. $\mathfrak{b}(\sqsubseteq) \geqslant \aleph_1$.

Tree relational system

Tree relational systems fit the setting of the First Preservation Theorem. (In the First Preservation Theorem, the codomain can be changed to an analytic set in some Polish space.)
Moreover, since each $\{x \in \lim T : x \sqsubset_k^- y\}$ is clopen, we have

$$P \text{ preserves } \sqsubseteq \iff P \text{ almost preserves } \sqsubseteq.$$

Tree relational system: definition of R^+

For a tree relational system $R = (\lim T, Y, \sqsubseteq)$, define $R^+ = (\lim T, Y, \sqsubseteq^+)$, where

$$\begin{aligned} x \sqsubseteq^+_n y &\iff \exists k \geqslant n \; x \sqsubseteq^-_k y, \\ x \sqsubseteq^+ y &\iff \forall n \; x \sqsubseteq^+_n y \\ &(\iff \exists^\infty k \; x \sqsubseteq^-_k y). \end{aligned}$$

Note the easy observation: $R \leqslant_{\mathrm{Tukey}} R^+$, in particular $\mathfrak{d}(R) \leqslant \mathfrak{d}(R^+)$. We want the opposite direction.

Tree relational system: a sufficient condition

Definition ($(\star)_R$)

Let $(\star)_R$ be the following statement: for every sufficient large θ and every countable $N \prec H_\theta$ with $R \in N$, we have y R-dominates N iff y R^+-dominates N for every $y \in Y$.

Lemma

Assume that ZFC proves $(\star)_R$. Then, if P almost preserves R, then P almost preserves R^+.

Tree relational system: a sufficient condition

Def. $(\star)_R :\Longleftrightarrow$ for every sufficient large θ and every countable $N \prec H_\theta$ with $R \in N$, we have y R-dominates N iff y R^+-dominates N for every $y \in Y$.

Lemma Assume that ZFC proves $(\star)_R$. Then, if P almost preserves R, then P almost preserves R^+.

Proof. Let $N \prec H_\theta$ countable with $P, \sqsubseteq \in N$. Let y be a R^+-dominating over N and $p \in P \cap N$. Since P almost preserves R, we can take N-generic $q \leqslant p$ forcing y is a R-dominating over $N[\dot{G}]$. By $(\star)_R$ applied in $V[G]$, q also forces y is a R^+-dominating over $N[\dot{G}]$. $\square$

Summary up to this point

Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration and $\mathfrak{x}$ be a cardinal invariant. In order to prove $P_{\omega_2} \Vdash \mathfrak{x} = \aleph_1$, it is sufficient to find a tree relational system $R = (\lim T, Y, \sqsubseteq)$ such that:

1. $\mathfrak{d}(R^+) = \mathfrak{x}$, provably.
2. $(\star)_R$, provably.
3. Each iterand of the iteration almost preserves R.

Examples

An example associated with $\mathfrak{b}$

Let $T = \omega^{<\omega}$, $Y = \omega^\omega$. For $x \in \lim T = \omega^\omega$ and $y \in \omega^\omega$, let $x \sqsubseteq_k^- y$ iff $x(k) < y(k)$.
Then $R = (\lim T, Y, \sqsubseteq)$ is a tree relational system.
It can be easily seen that $R^+ \equiv_{\mathrm{Tukey}} (\omega^\omega, \leqslant^*)^\perp$. So $\mathfrak{d}(R^+) = \mathfrak{b}$.

Claim $(\star)_R$ holds.

Proof. Let N be a countable elementary submodel and $y \in \omega^\omega$ R-dominates N. We claim that $y \in \omega^\omega$ R^+-dominates N (that is, y is an unbounded real over N). Let $x \in \omega^\omega \cap N$ and $n \in \omega$. We must find $n' \geqslant n$ such that $x(n') < y(n')$. Consider $x' \in \omega^\omega$ defined by $x' = (y \restriction n) \cup (x \restriction [n, \omega))$, which is in N. Since y R-dominates N, we can find n' such that $x'(n') < y(n')$. But this n' must be $\geqslant n$. □

An example associated with $\mathrm{cov}(\mathcal{N})$ (1/2)

Let

$$\begin{aligned}\mathrm{SC} = \{(\bar{I}, \varphi) : {} & \bar{I} \text{ is an interval partition of } \omega, \\ & \varphi \in \prod_n \mathcal{P}(2^{I_n}) \text{ and } \frac{|\varphi(n)|}{2^{|I_n|}} \leqslant 2^{-n-1} \text{ for all } n\}.\end{aligned}$$

For $(\bar{I}, \varphi) \in \mathrm{SC}$, the set

$$\{x \in 2^\omega : (\exists^\infty n) x \restriction I_n \in \varphi(n)\}$$

is called a small set.

An example associated with $\mathrm{cov}(\mathcal{N})$ (2/2)

Consider the tree relational system $R = (2^\omega, \mathrm{SC}, \sqsubseteq)$, where

$$x \sqsubseteq_k^- (\bar{I}, \varphi) \iff x \restriction I_k \in \varphi(k).$$

Note that $\mathfrak{d}(R^+) = \mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}(R^+) \geqslant \aleph_1$ (The former follows from Bartoszyński's theorem stating every null set is covered by 2 small sets; the latter follows from an easy observation that every countable subset of 2^ω is covered by a small set).

It can be also checked that this R satisfies $(\star)_R$ (by using finite modifications).

An example associated with non($\mathcal{M}$)

(This example is already in Goldstern's paper and Bartoszyński–Judah book.)
Take an enumeration $2^{<\omega} = \{s_k : k \in \omega\}$. Set

$$D = \{f : \omega \to 2^{<\omega} : (\forall k \in \omega) s_k \subseteq f(k)\}.$$

Let $f \sqsubseteq_k^- y \iff y \in [f(k)]$. Consider the tree relational system $R = (D, 2^\omega, \sqsubseteq)$.

It can be checked that $\mathfrak{d}(R^+) = \mathrm{non}(\mathcal{M})$, $\mathfrak{b}(R^+) = \mathrm{cov}(\mathcal{M})$ and R satisfies $(\star)_R$.

The example associated with non($\mathcal{E}$)

Let $\bar{\epsilon} = \langle \epsilon_k : k \in \omega \rangle \in (\mathbb{R}_{>0})^\omega$ and assume $\liminf_k \epsilon_k = 0$. Let $\Omega_{\bar{\epsilon}} = \{\langle c_n : n \in \omega \rangle$: each c_n is clopen subset of 2^ω and $\mu(c_n) \leqslant \epsilon_n$ $\}$. Let $\bar{c} \sqsubseteq_k^- y \iff y \notin c_k$. Consider the tree relational system $R = (\Omega_{\bar{\epsilon}}, 2^\omega, \sqsubseteq)$.

It can be checked that $\mathfrak{d}(R^+) = \text{non}(\mathcal{E})$, $\mathfrak{b}(R^+) = \text{cov}(\mathcal{E})$ and R satisfies $(\star)_R$.

Consistency results

Consequence of goodness

Theorem (G. and Mejía)

Assume $(\star)_R$. If P is proper and $(R^+)^\perp$-good, then it almost preserves R.

Proof. Let N be a countable elementary submodel, $P, R \in N$, $p \in P \cap N$ and $y \in Y$ be R-dominating over N. By $(\star)_R$, y is also R^+-dominating over N. Let $\dot{x} \in N$ be a P-name of an element in $\lim T$. Then by goodness, there is a countable nonempty subset $H \subseteq \lim T$ such that for every R-dominating $z \in Y$ over H, we have $P \Vdash \dot{x} \sqsubseteq^+ z$. Since $H \subseteq N$, we have $P \Vdash \dot{x} \sqsubseteq^+ y$. Thus $\Vdash$ "y is an R-dominating real over $N[\dot{G}]$". So any N-generic condition $q \leqslant p$ forces this.

$\mathbf{PT}_H$ almost preserves all tree relational systems

Let $H \in \omega^\omega$. $\mathbf{PT}_H$ is a forcing notion, ordered by $\subseteq$, whose conditions are subtrees $p \subseteq \bigcup_n \prod_{i<n} H(i)$ such that

1. For every $t \in p$, we have $|\mathrm{succ}_p(t)| = 1$ or $|\mathrm{succ}_p(t)| = H(|t|)$.
2. For every $t \in p$, there is $s \geqslant t$ in p such that $|\mathrm{succ}_p(s)| = H(|s|)$.

Theorem (G. and Mejía)

$\mathbf{PT}_H$ almost preserves any tree relational system $R = (\lim T, Y, \sqsubseteq)$.

Lemma

Let N be a countable elementary submodel, $H, R \in N$, $p \in \mathbf{PT}_H \cap N$, $D \in N$ dense open subset of $\mathbf{PT}_H$, $\dot{x} \in N$ be a $\mathbf{PT}_H$-name of a real in $\lim T$ and $n \in \omega$. Assume that $y \in Y$ R-dominates N. Then there is $p' \leqslant_n p$ in N such that $p' \Vdash \dot{x} \sqsubseteq y$ and $\forall t \in \mathrm{split}_{n+1}(p')\ p' \wedge t \in D$.

$\mathbf{PT}_H$ almost preserves all tree relational systems

Lemma Let N be a cem, $H, R \in N$, $p \in \mathbf{PT}_H \cap N$, $D \in N$ dense open subset of $\mathbf{PT}_H$, $\dot{x} \in N$ be a $\mathbf{PT}_H$-name of a real in $\lim T$ and $n \in \omega$. Assume that $y \in Y$ R-dominates N. Then there is $p' \leqslant_n p$ in N such that $p' \Vdash \dot{x} \sqsubseteq y$ and $\forall t \in \mathrm{split}_{n+1}(p')$ $p' \wedge t \in D$.

Proof. Without loss of generality, we assume that p decides $\dot{x} \restriction k$ at $\mathrm{split}_{n+k+1}(p)$ for each k.
Pick $\langle z_t : t \in \mathrm{split}_{n+1}(p)\rangle \in N$ such that $z_t \in \lim p$ extends t. This z_t gives an interpretation $x_t \in \lim T$ of $\dot{x}$, even $\langle x_t : t \in \mathrm{split}_{n+1}(p)\rangle \in N$. Thus we have $x_t \sqsubseteq y$. Pick $k_t \in \omega$ such that $x_t \sqsubset^-_{k_t} y$. Since $\sqsubseteq^-_{k_t}$ is open, we can take $l_t \in \omega$ such that $[x_t \restriction l_t] \subseteq \{x \in \lim T : x \sqsubset^-_{k_t} y\}$. This $x_t \restriction l_t$ is decided by some $s_t \in P$ with $t \subseteq s_t \subseteq z_t$. Pick $p'_t \leqslant p \wedge s_t$ in D and let $p' = \bigcup_{t \in \mathrm{split}_{n+1}(p)} p'_t$. By finiteness of $\mathrm{split}_{n+1}(p)$, we have $p' \in N$. □

$\mathbf{PT}_H$ almost preserves all tree relational sys

Lemma Let N be a cem, $H, R \in N$, $p \in \mathbf{PT}_H \cap N$, $D \in N$
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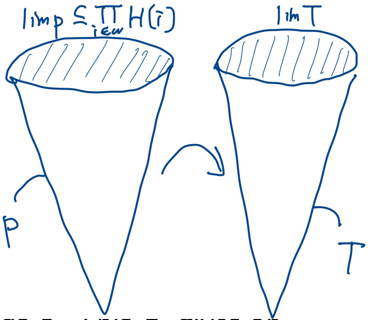

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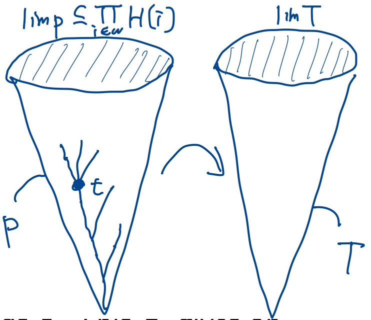

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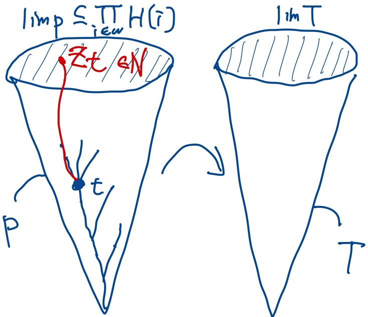

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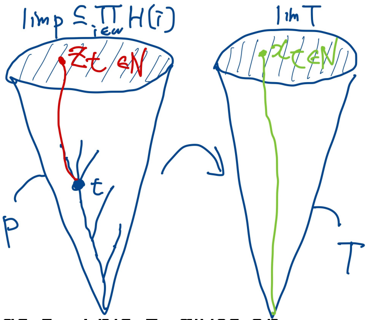

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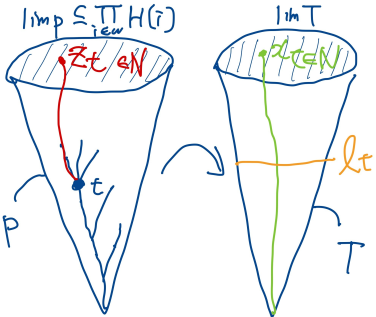

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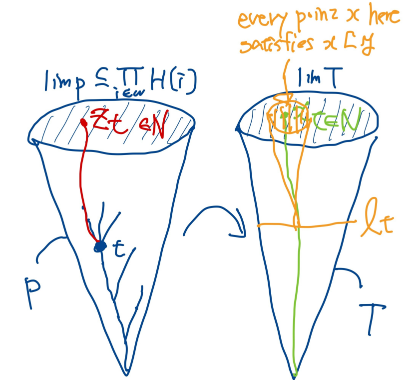

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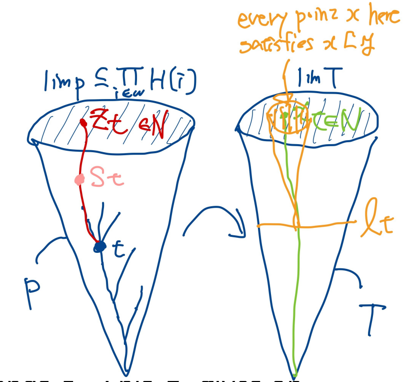
$\mathbf{PT}_H$ almost preserves all tree relational sys



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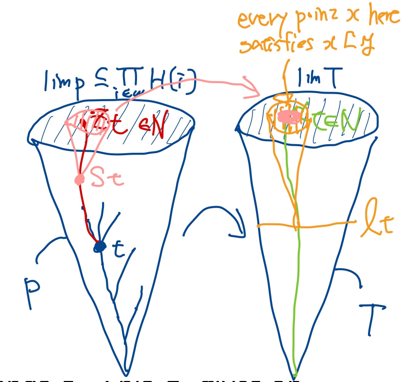

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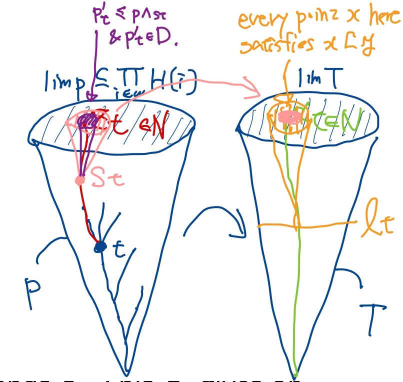

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$\mathbf{PT}_H$ almost preserves all tree relational systems

Note that the same argument works for every finitely branching limsup creature forcing.

A corollary

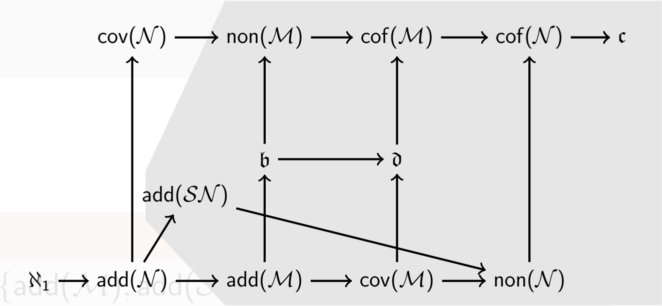
Corollary

It is consistent that $\mathrm{cov}(\mathcal{N}) < \min\{\mathrm{add}(\mathcal{M}), \mathrm{add}(\mathcal{SN})\}$.

Proof. Iterate $\mathbf{PT}_H$ and the Hechler forcing alternatively, bookkeeping H. □

This result can be strengthened to $\mathrm{supcov} < \min\{\mathrm{add}(\mathcal{M}), \mathrm{add}(\mathcal{SN})\}$.

A corollary



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$\mathbf{PT}_{f,g}$ preserves $\mathrm{non}(\mathcal{E})$

$\mathbf{PT}_{f,g}$ a well-known proper forcing notion that increases $\mathrm{non}(\mathcal{M})$.

Theorem (G. and Mejía)

$\mathbf{PT}_{f,g}$ preserves the tree relational system associated with $\mathrm{non}(\mathcal{E})$.

This result seems interesting because the eventually different real forcing (a ccc forcing that increases $\mathrm{non}(\mathcal{M})$) also increases $\mathrm{non}(\mathcal{E})$ (see Cardona's paper in 2024).

Corollary

It is consistent that $\max\{\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{E}), \mathfrak{d}\} < \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$.

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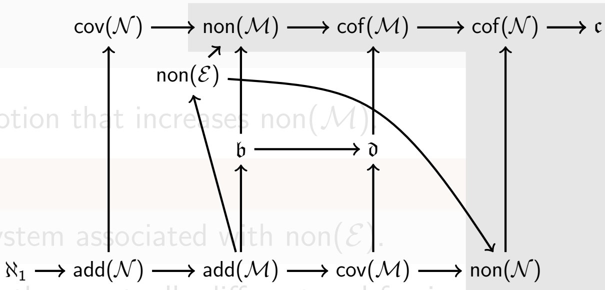

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A conjecture

There are 36 many assignments of $\aleph_1$ and $\aleph_2$ to the cardinal invariants appearing in Cichoń's diagram and $\mathrm{cov}(\mathcal{E})$ and $\mathrm{non}(\mathcal{E})$ not violating currently known ZFC results. (We checked this number by a computer program).

We conjecture that all of them are forceable.

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