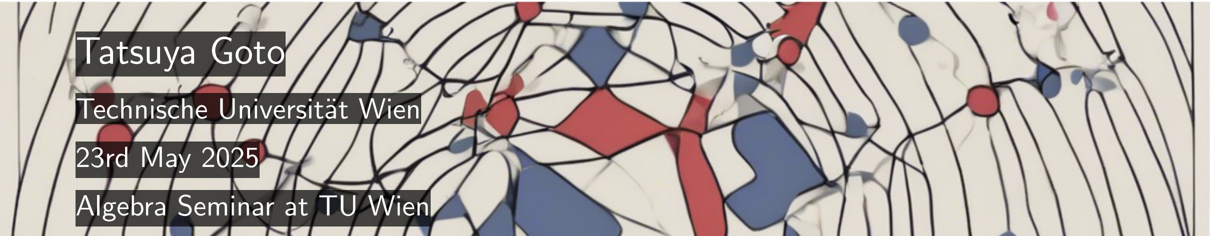
Preservation of $\mathcal{E}$-positivity along some proper forcing notions



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Cardinal invariants

Cardinal invariants are cardinals defined in terms of propeties of real numbers. ZFC proves many of them lie between $\aleph_1$ and $2^{\aleph_0}$, but the actual values are not decided.

Example: $\mathrm{add}(\mathcal{N})$ is the minimum cardinal κ such that a union of κ-many Lebesgue null sets is not Lebesgue null.
$\mathrm{cov}(\mathcal{N})$ is the number how many Lebesgue null sets are needed to cover the real line.
$\mathrm{non}(\mathcal{N})$ is the minimum cardinality of a non-null set.

The researchers of this field want to clarify the properties of the real line through studying cardinal invariants.

Cichoń's diagram

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

ccc fsi vs proper csi

The two most notable methods separating cardinal invariants are **ccc fsi** (coutable chain condition finite support iterations) and **proper csi** (proper countable support iterations).
Although ccc fsi is the older method, it has gained renewed value in the last decade as Cichoń's maximum has been constructed using this method.

The demerit of ccc fsi: they always force that $\mathrm{non}(\mathcal{M}) \leqslant \mathrm{cov}(\mathcal{M})$.

The demerit of proper csi: they always force that $\mathfrak{c} \leqslant \aleph_2$.

The ideal $\mathcal{E}$

In recent years, it is common to add cardinal invariants $\mathrm{cov}(\mathcal{E})$ and $\mathrm{non}(\mathcal{E})$ defined by the ideal $\mathcal{E}$ into Cichoń's diagram and consider them (For example results in 2024 due to Cardona, Cardona–Mejía–Uribe and Yamazoe).
Here,

$$\mathcal{E} = \{A \subseteq 2^\omega : (\exists \langle F_n : n \in \omega \rangle)[A \subseteq \bigcup_n F_n, \text{ each } F_n \text{ is closed and has Lebesgue measure } 0]\}.$$

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[rr] \arrow[rrrrddd, bend right] & & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[rr] & & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathrm{non}(\mathcal{E}) \arrow[ur] \arrow[rrrrddd, bend left] & & & & & \\
 & & & \mathfrak{b} \arrow[r] \arrow[uu] & \mathfrak{d} \arrow[uu] & & & \\
 & & & & & \mathrm{cov}(\mathcal{E}) \arrow[luuu] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[rr] \arrow[uuuu] & & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[uu] \arrow[luuu] & \mathrm{cov}(\mathcal{M}) \arrow[rr] \arrow[uu] \arrow[ru] & & \mathrm{non}(\mathcal{N}) \arrow[uuuu] &
\end{tikzcd}$$

Facts about $\mathcal{E}$

Easily we can see

$$\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}.$$

Thus,

$$\mathrm{non}(\mathcal{E}) \leqslant \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\} \text{ and}$$

$$\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} \leqslant \mathrm{cov}(\mathcal{E}).$$

Moreover, we have

$$\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leqslant \mathrm{non}(\mathcal{E}) \leqslant \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\} \text{ and}$$

$$\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} \leqslant \mathrm{cov}(\mathcal{E}) \leqslant \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}.$$

Consistency results about $\mathcal{E}$

Bartoszyński and Shelah proved:

$$\mathrm{non}(\mathcal{E}) < \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\} \text{ is consistent.}$$

Moreover,

$$\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} < \mathrm{cov}(\mathcal{E}) \text{ is consistent.}$$

Each of them is obtained by the finite support iteration of the random forcing (of length ω_2 and $\omega_2 + \omega_1$, respectively).

We want to separate them by countable support iterations!

We want to separate the cardinal invariants of $\mathcal{E}$ from other cardinal invariants even by countable support iterations!

$\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} < \mathrm{cov}(\mathcal{E})$ by countable support iterations

Forcing $\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} < \mathrm{cov}(\mathcal{E})$ by a countable support iteration was done by Sabok and Zapletal using the idealized forcing $P_{\mathcal{E}}$.

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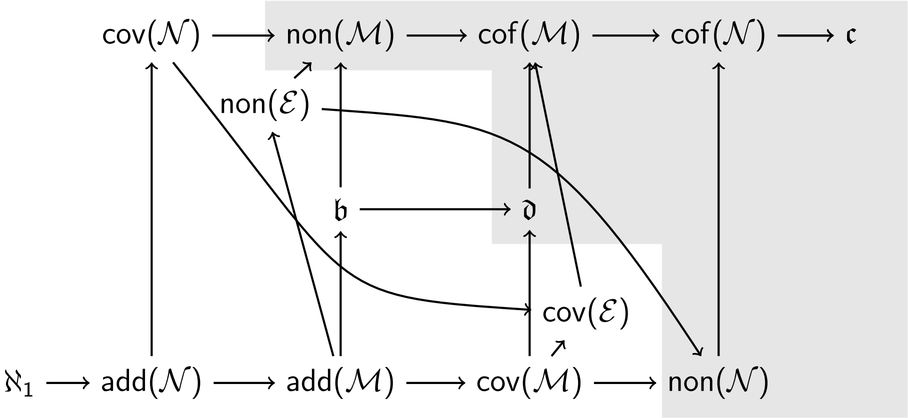
Our results

By a countable support iteration, we have $\mathrm{non}(\mathcal{E}) < \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$.
In detail, we have the model by a countable support iteration separating as shown in the following figure (under large cardinals):



Forcing notions we use

Forcing notions we use are the following three:

1. $\mathbf{PT}_{f,g}$ (increasing $\mathrm{non}(\mathcal{M})$)
2. Miller forcing $\mathbf{M}$ (increasing $\mathfrak{d}$)
3. $\mathbf{S}_{g,g*}$ (increasing $\mathrm{non}(\mathcal{N})$)

The preservation theorem we use

Theorem (Zapletal) (under a large cardinal assumption)

Suppose P is a universally Baire proper real forcing. If P preserves $\mathcal{E}$-positivity of Borels sets, then countable support iterations of P of arbitrary length also preserve it.

Here, that P preserves $\mathcal{E}$-positivity of Borels sets means $P \Vdash \dot{B} \cap V \notin \mathcal{E}$ for every Borel $\mathcal{E}$-positive set B.
Actually, he states this theorem for σ-ideals σ-generated by a universally Baire collection of closed sets.
We think if we strengthen the assumption on P then we can remove the large cardinal assumption as follows:

Conjecture (ZFC) Suppose $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ proper ideal. If $P_{\mathcal{I}}$ preserves $\mathcal{E}$-positivity of Borels sets, then countable support iterations of $P_{\mathcal{I}}$ also preserve it.

What we showed

The following three forcing notions preserve $\mathcal{E}$-positivity.

1. $\mathbf{PT}_{f,g}$ (increasing $\mathrm{non}(\mathcal{M})$)
2. $\mathbf{M}$ (increasing $\mathfrak{d}$)
3. $\mathbf{S}_{g,g*}$ (increasing $\mathrm{non}(\mathcal{N})$)

An easy lemma

Lemma

Let (X, μ) be a probability space.

1. Let $\langle A_j : j < N \rangle$ be a sequence of measurable subsets of X such that $\mu(A_j) \leqslant a$ (for every $j < N$). Then for $0 < M \leqslant N$, we have

$$\mu(\{x \in X : |\{j < N : x \in A_j\}| \geqslant M\}) \leqslant \frac{aN}{M}.$$

2. Let $\langle B_j : j < N \rangle$ be a sequence of measurable subsets of X such that $\mu(B_j) \geqslant b$ (for every $j < N$). Then for every $M < N$, we have

$$\mu(\{x \in X : |\{j < N : x \in B_j\}| \geqslant M\}) \geqslant 1 - \frac{(1-b)N}{N-M}.$$

$\mathbf{PT}_{f,g}$: definition

Let $f \in \omega^\omega$ and $g \in \omega^{\omega \times \omega}$ be functions satisfying the following conditions:

1. For all $n \in \omega$, it holds that $f(n) > \prod_{j<n} f(j)$.
2. For all $n, j \in \omega$, we have $g(n, j+1) > f(n) \cdot g(n, j)$.
3. $\min\{j \in \omega : g(n, j) > f(n+1)\} \to \infty$ (as $n \to \infty$).

Definition ($\mathbf{PT}_{f,g}$)

Let $T \in \mathbf{PT}_{f,g}$ iff the following conditions hold:

1. T is a perfect subtree of $\bigcup_{n\in\omega} \prod_{i<n} f(i)$.
2. $\lim_{n\to\infty} \min\{\mathrm{nor}_n(\mathrm{succ}_T(s)) : s \in T \cap \omega^n\} = \infty$.

Here, $\mathrm{nor}_n(A) = \max\{m : |A| \geqslant g(n, m)\}$ for $A \subseteq \omega$. The order on $\mathbf{PT}_{f,g}$ is defined by: $T \leqslant S \Leftrightarrow T \subseteq S$ for $S, T \in \mathbf{PT}_{f,g}$.

$\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity

Theorem (containing errors!)

$\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity.

$\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (1/6)

Theorem $\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity.

This proof is based on the proof that $\mathbf{PT}_{f,g}$ preserves non($\mathcal{N}$). Let $A \subseteq 2^\omega$ be such that $A \notin \mathcal{E}$. Take $T \in \mathbf{PT}_{f,g}$ such that $T \Vdash A \in \mathcal{E}$. Then there is $\langle \dot{I}_n, \dot{J}_n : n \in \omega \rangle$ such that $T \Vdash A \subseteq \{x \in 2^\omega : (\forall^\infty n) x \restriction \dot{I}_n \in \dot{J}_n\}$, $\Vdash$ "$\langle \dot{I}_n : n \in \omega \rangle$ is an interval partiton" and $\Vdash \dot{J}_n \subseteq 2^{\dot{I}_n}, |\dot{J}_n|/2^{|\dot{I}_n|} < 2^{-n}$ for all n.

Since $\mathbf{PT}_{f,g}$ is ω^ω-bounding, we may assume that there is $\langle I_n : n \in \omega \rangle$ in the ground model such that $\Vdash \dot{I}_n = I_n$. Shrinking the condition, we may assume that there is an increasing sequence $\langle k_n : n \in \omega \rangle$ such that

$$(\forall n)(\forall s \in T \cap \omega^{k_n})(T^s \text{ decides the values of } \dot{J}_0, \ldots, \dot{J}_n).$$

For $s \in T$ with $|s| \geqslant k_n$ and $j \leqslant n$, we let J_j^s be the decided value of $\dot{J}_n$ by T^s. Otherwise we let $J_j^s = \varnothing$.

$\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (2/6)

Put

$$P = \prod_{n=0}^{\infty} \left(1 - \frac{1}{f(n)}\right).$$

Take an infinite subset H of ω such that $\prod_{n \in H}(1 - 2^{-n}) > 1 - \frac{P}{2}$ (Note that infinite product $\prod_{n \in \omega}(1 - 2^{-n})$ converges!).

$\mathbf{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (3/6)

Put

$$r(n) = \min\{\mathrm{nor}_n(\mathrm{succ}_T(s)) : s \in T \cap \omega^n\} \text{ and } s_0 = \mathrm{stem}(T).$$

For $s \in T \cap \omega^{k_n - 1}$ we put

$$X_n^s = \{t : I_{\leqslant n} \to 2 : (\exists A \subseteq \mathrm{succ}_T(s))(\mathrm{nor}_{|s|}(A) \geqslant r(|s|) - 1 \text{ and } (\forall a \in A)(\forall m \in [1, n] \cap H)(t \restriction I_m \in 2^{I_m} \smallsetminus J_m^{s^\frown \langle a \rangle})\}.$$

By the downward induction, for $s \in T \cap \omega^{<(k_n - 1)}$, we put

$$X_n^s = \{t : I_{\leqslant n} \to 2 : (\exists A \subseteq \mathrm{succ}_T(s))(\mathrm{nor}_{|s|}(A) \geqslant r(|s|) - 1 \text{ and } (\forall a \in A)(t \in X_n^{s^\frown \langle a \rangle})\}.$$

Induce the uniform measure μ_n on $2^{I_{\leqslant n}}$. Then, for every $s \in T \cap \omega^{k_n - 1}$, we have $\mu_n(2^{I_{n+1}} \smallsetminus J_{n+1}^{s^\frown \langle a \rangle}) \geqslant \prod_{m \in [1,n] \cap H} (1 - 2^{-m}) =: 1 - a_n$.

$\mathrm{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (4/6)

Using the previous lemma (2) applied to $2^{I_n} \setminus J_n^{s^\frown\langle a\rangle}$ (for $a \in \mathrm{succ}_T(s)$) and $M := g(|s|, r(|s|) - 1)$, we also have, for every $s \in T \cap \omega^{k_n - 2}$,

$$\mu_n(X_n^s) \geqslant 1 - \frac{1 - a_n g(|s|, r(|s|))}{g(|s|, r(|s|)) - g(|s|, r(|s| - 1))} \geqslant 1 - a_n \left(1 - \frac{1}{f(k_n - 1)}\right)^{-1}.$$

By the downward induction, for $s \in T \cap \omega^{k_{n+1} - l}$, we have

$$\mu_n(X_n^s) \geqslant 1 - a_n \left(1 - \frac{1}{f(k_n - 1)}\right)^{-1} \cdots \left(1 - \frac{1}{f(k_n - l)}\right)^{-1}.$$

Thus we have

$$\mu_n(X_n^s) \geqslant 1 - \frac{a_n}{\prod_{l=0}^{\infty} \left(1 - \frac{1}{f(l)}\right)}.$$

Then we put

$$A^* = \{x \in 2^\omega : (\exists^\infty n)(x \restriction I_{\leqslant n} \in X_n^{s_0})\}.$$

$\mathrm{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (5/6)

We have

$$\mu(A^*) \geqslant \limsup_{n\to\infty} \mu(X_n) \geqslant 1 - \frac{\lim_{n\to\infty} a_n}{P} = 1 - \frac{1 - \prod_{n\in H}(1-2^{-n})}{P} \geqslant \frac{1}{2}.$$

Also we know A^* is a G_δ set and A^* is a tail set. Therefore A^* is a co-$\mathcal{E}$ set. Then we can take $x \in A \cap A^*$.

Consider $n \in \omega$ such that $x \restriction I_n \in X_n^{s_0}$. Then

$$(\exists A_0 \subseteq \mathrm{succ}_T(s_0))(\mathrm{nor}_{|s_0|}(A) \geqslant r(|s_0|) - 1, (\forall a \in A)(x \restriction I_n \in X_n^{s^\frown\langle a\rangle})).$$

Shrink the successors of s_0 to this A_0. Next for each $a_0 \in A_0$, since $x \restriction I_n \in X_n^{s^\frown\langle a\rangle}$, we have

$$(\exists A_1 \subseteq \mathrm{succ}_T(s_0{}^\frown\langle a_0\rangle))(\mathrm{nor}_{|s_0|+1}(A) \geqslant r(|s_0|+1) - 1, (\forall a_1 \in A_1)(x \restriction I_n \in X_n^{s^\frown\langle a_0, a_1\rangle})).$$

Shrink the successors of $s_0{}^\frown\langle a_0\rangle$ to this A_1.

$\mathrm{PT}_{f,g}$ preserves $\mathcal{E}$-positivity (6/6)

Repeating this process $k_{n+1} - |s_0|$ times we get

$$(\exists \mathbf{t}_n \subseteq T \cap \omega^{\leqslant k_n} \text{ subtree})[(\forall s \in \mathbf{t}_n)\, \mathrm{nor}_{|s|}(\mathrm{succ}_{\mathbf{t}_n}(s)) \geqslant r(|s|) - 1, \text{ and}$$
$$(\forall s \in \mathbf{t}_n \cap \omega^{k_n})(\forall m \in [1, n] \cap H)(x \restriction I_{\leqslant m} \in 2^{I_{\leqslant m}} \smallsetminus J^s_m)].$$

Using a compactness argument, we can amalgamate these finite trees $\langle \mathbf{t}_n : n \in \omega, x \restriction I_{\leqslant n} \in X^{s_0}_n \rangle$ to get a condition T'. This T' forces that $(\exists^\infty n)(x \restriction I_n \notin \dot{J}_n)$, which is contradiction to $T' \leqslant T$. (QED)

Under large cardinals:

$$
\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[rr] \arrow[dddrrrr, bend right] & & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[rr] & & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
& & \mathrm{non}(\mathcal{E}) \arrow[ur] \arrow[dddrrrr, bend left] & & & & & \\
& & & \mathfrak{b} \arrow[uu] \arrow[r] & \mathfrak{d} \arrow[uu] & & & \\
& & & & & \mathrm{cov}(\mathcal{E}) \arrow[uuul] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[rr] \arrow[uuuu] & & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[uu] \arrow[uuul] & \mathrm{cov}(\mathcal{M}) \arrow[rr] \arrow[uu] \arrow[ur] & & \mathrm{non}(\mathcal{N}) \arrow[uuuu] &
\end{tikzcd}
$$

Future works

⓪ **We have to fix errors in the proof.**

① We would like to remove the large cardinal assumption.

② Fischer–Goldstern–Kellner-Shelah constructed a model satisfying $\mathfrak{d} = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}$ using techniques of countable support products. We would like to add $\mathrm{non}(\mathcal{E})$ to their separation.

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