博士論文

Goldstern's principle and other applications of cardinal invariants of the continuum

(ゴールドスターンの原理と連続体の基数不変量の他の応用)

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Chapter 1

Introduction

The subject of this thesis is cardinal invariants and their application to problems in other fields of mathematics.

In the first place, set theory originates from the naive action of counting numbers and studies what happens when we extend this notion to infinite sets. The notion that extends the number of objects to infinite sets is cardinality, which is the most important notion in set theory. The number concept that is used for cardinality is the cardinal.

Cantor [Can74] proved in 1874 that the cardinality of the real line is strictly bigger than the cardinality of the natural numbers, thus giving birth to the field of set theory. Cantor subsequently made a conjecture called the continuum hypothesis, which states that there are no other cardinalities strictly between the cardinality of the natural numbers and the cardinality of the real line, but he was unable to solve it. Eventually, it was shown by Gödel [Göd38] in 1938 that the negation of the continuum hypothesis cannot be proved from the standard axiomatic system of set theory, ZFC, and by Cohen [Coh63] in 1963 that the continuum hypothesis cannot be proved from ZFC. For this we say that the continuum hypothesis is independent from ZFC. The method of forcing, which Cohen invented to obtain this consistency, is a very big breakthrough in set theory.

An outcome of the invention of the forcing method is that we are able to prove various consistency results by distinguishing finely notions such as Lebesgue measurability, Baire category and the growth of functions from natural numbers to themselves, etc. The tools separating such notions are the cardinal invariants, which are definable cardinals that we extract from the properties of the real line. The cardinal invariants concerning the 3 notions, Lebesgue measurability, Baire category and the growth of functions have been well studied and Cichoń's diagram (see Chapter 2) summarizes this study. In 1993, Cichoń's diagram was proved to be complete [BJS93], i.e., no more lines can be drawn. Recently, an impressive result has been achieved in the form of Cichoń's maximum [GKS19], a model that separates the cardinal invariants of Cichoń's diagram simultaneously as much as possible.

However, there are many cardinal invariants other than these classical cardinal invariants and there is the possibility of finding applications that have not yet been found for classical cardinal invariants. This thesis explores these possibilities.

In Chapter 3, we study Goldstern's principle stating the union of continuum many null sets is also null under some assumptions. The assumptions are definability and monotonicity of the family. Goldstern proved the principle for the definability assumption of analytic sets. The proof is interesting in that it uses the forcing method normally used for consistency proofs to show a theorem in ZFC.

We examined this principle from various aspects and proved a number of interesting results. One of the main results in this chapter is to improve on Goldstern's proof and show that the principle holds for sets that are not analytic but coanalytic. Another important result is that the principle obtained by completely removing the condition on definability is independent from ZFC. "Goldstern's principle about unions of null sets" (`https://arxiv.org/abs/2206.08147`) is a preprint that has been submitted for publication and is available on arXiv. The content of Chapter 3 of this thesis coincides with that of this preprint.

In Chapter 4, we discuss cardinal invariants of Hausdorff measures. Hausdorff measures, as the name implies, were first conceived by Hausdorff and have been classically well studied as measures where parameters can vary to obtain many measures finer than the Lebesgue measure, and they are a fundamental tool in the field of fractal geometry. Hausdorff measures are important mathematical objects that differ from the above three classical notions so that it is meaningful to add cardinal invariants of them to Cichoń's diagram and consider them. The main result in this chapter is that we can make a lot of cardinal invariants of Hausdorff measures take different values. " Cardinal invariants associated with Hausdorff measures" (`https://arxiv.org/abs/2112.07952`) is a preprint that has been submitted for publication and is available on arXiv. The content of Chapter 4 of this thesis coincides with that of this preprint.

In Chapter 5, we study Keisler's theorem. The theorem by Keisler and Shelah saying that the notion of elementary equivalence can be characterized in terms of ultrapowers is a milestone in model theory. However, Keisler's proof assumes the general continuum hypothesis, and Shelah's proof uses ultrapowers with a fairly large index set, so there is still room for further research. How does the Keisler-Shelah theorem behave with ultrapowers using relatively small index sets without assuming the general continuum hypothesis? The main result in this chapter is that Keisler's theorem and many other related principles are related to the cardinal invariant $\mathrm{cov}(\mathcal{M})$ but $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$ is consistent with a version of Keisler's theorem. "Keisler's theorem and cardinal invariants" was published in Journal of Symbolic Logic, Volume 89(2) (pp. 905-917) (`https://doi.org/10.1017/jsl.2022.77`). Also "Keisler's theorem and cardinal invariants at uncountable cardinals" was published in RIMS Kôkyûroku No.2290: Large Cardinals and the Continuum (`https://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/2290-05.pdf`). The content of Chapter 5 of this thesis contains that of these two papers.

In Chapter 6, we discuss cardinal invariants associated with the notion of comparability and incomparability of posets. We show in this chapter that for many well-known posets, comparability numbers and incomparability numbers often coincide with existing cardinal invariants. "The Comparability Numbers and the Incomparability Numbers" was published online in Order (`https://doi.org/10.1007/s11083-024-09672-y`). The content of Chapter 6 of this thesis coincides with that of this paper.

In Chapter 7, we study cardinal invariants defined from a game-theoretic viewpoint. We show that a game-theoretic interpretation of classically well-studied cardinal invariants yields new cardinal invariants, and we investigate the relationship between these cardinal invariants. The main result is the cardinal invariants obtained by considering a new game, the so-called splitting game, consistently differ from any of classical cardinal invariants. "Game-theoretic variants of cardinal invariants" is a preprint written by Jorge Antonio Cruz Chapital, the author and Yusuke Hayashi and has been submitted for publication and is available on arXiv (`https://arxiv.org/abs/2308.12136`). The content of Chapter 7 of this thesis contains that of this preprint. "Game-theoretic variants of splitting number" is a preprint written by Jorge Antonio Cruz Chapital, the author, Yusuke Hayashi and Takashi Yamazoe and has been submitted for publication and is available on arXiv (`https://arxiv.org/abs/2412.19556`). Chapter 7 of this thesis and this preprint share some contents.

Chapter 2

Preliminaries

The axiomatic framework for the most of the discussion in this thesis is ZFC, that is Zermelo–Fraenkel set theory with the axiom of choice.

The set theory notation used in this thesis is standard following [Kun14] and [Jec03].

$(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations to say "for all but finitely many n" and "there exist infinitely many n", respectively.

For $A, B \subseteq \omega$, the relation $A \subseteq^* B$ means that $A \setminus B$ is finite. We say B almost contains A if $A \subseteq^* B$ holds. In addition, for $x, y \in \omega^\omega$, the relation $x \leq^* y$ means $(\forall^\infty n)(x(n) \leq y(n))$. We say y dominates x if $x \leq^* y$ holds. We sometimes use the following totally domination relation: $x \leq y$, which means $(\forall n \in \omega)(x(n) \leq y(n))$ for $x, y \in \omega^\omega$. We use the notation $x <^\infty y$, that means $\neg(y \leq^* x)$.

$\mathfrak{c}$ denotes the cardinality of the continuum.

$\mathcal{M}$ denotes the all meager subsets of 2^ω. And also, $\mathcal{N}$ denotes the all Lebesgue null subsets of 2^ω.

The following are some standard cardinal invariants.

Definition 2.0.1. (1) $\mathcal{A} \subseteq \omega^\omega$ is a dominating family if for every $x \in \omega^\omega$, there is $y \in \mathcal{A}$ that dominates x. Define the dominating number $\mathfrak{d}$ by $\mathfrak{d} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \omega^\omega \text{ a dominating family}\}$.

(2) $\mathcal{A} \subseteq \omega^\omega$ is an unbounded family if for every $x \in \omega^\omega$, there is $y \in \mathcal{A}$ that is not dominated by x. Define the bounding number $\mathfrak{b}$ by $\mathfrak{b} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \omega^\omega \text{ an unbounded family}\}$.

(3) For $x \in \mathcal{P}(\omega)$ and $y \in [\omega]^\omega$, we say x splits y if both of $y \cap x$ and $y \setminus x$ are infinite. $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a splitting family if for every $y \in [\omega]^\omega$, there is $x \in \mathcal{A}$ such that x splits y. Define the splitting number $\mathfrak{s}$ by $\mathfrak{s} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ a splitting family}\}$.

(4) For $x \in \mathcal{P}(\omega)$ and $y \in [\omega]^\omega$, we say y reaps x if either $y \subseteq^* x$ or $y \subseteq^* \omega \setminus x$ holds. This is equivalent to say x does not split y. $\mathcal{A} \subseteq [\omega]^\omega$ is a reaping family if for every $x \in \mathcal{P}(\omega)$, there is $y \in \mathcal{A}$ such that y reaps x. Define the reaping number $\mathfrak{r}$ by $\mathfrak{r} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ a reaping family}\}$.

(5) $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a σ-splitting family if for every $f \in ([\omega]^\omega)^\omega$, there is $x \in \mathcal{A}$ such that x splits $f(n)$ for every $n \in \omega$. Define the σ-splitting number $\mathfrak{s}_\sigma$ by $\mathfrak{s}_\sigma = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ a } \sigma\text{-splitting family}\}$.

(6) $\mathcal{A} \subseteq [\omega]^\omega$ is a σ-reaping family if for every $f \in (\mathcal{P}(\omega))^\omega$, there is $y \in \mathcal{A}$ such that y reaps $f(n)$ for every $n \in \omega$. Define the σ-reaping number $\mathfrak{r}_\sigma$ by $\mathfrak{r}_\sigma = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ a } \sigma\text{-reaping family}\}$.

(7) For an ideal $\mathcal{I}$ on a set X: $\mathrm{add}(\mathcal{I})$ (the additivity number of $\mathcal{I}$) is the smallest cardinality of a family F of sets in $\mathcal{I}$ such that the union of F is not in $\mathcal{I}$.

(8) For an ideal $\mathcal{I}$ on a set X: $\mathrm{cov}(\mathcal{I})$ (the covering number of $\mathcal{I}$) is the smallest cardinality of a family F of sets in $\mathcal{I}$ such that the union of F is equal to X.

(9) For an ideal $\mathcal{I}$ on a set X: $\mathrm{non}(\mathcal{I})$ (the uniformity of $\mathcal{I}$) is the smallest cardinality of a subset A of X such that A does not belong to $\mathcal{I}$.

(10) For an ideal $\mathcal{I}$ on a set X: $\mathrm{cof}(\mathcal{I})$ (the cofinality of $\mathcal{I}$) is the smallest cardinality of a family F of sets in $\mathcal{I}$ that satisfies the following condition: for every $A \in \mathcal{I}$, there is $B \in F$ such that $A \subseteq B$.

Considering these four invariants $\mathrm{add}(\mathcal{I})$, $\mathrm{cov}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{I})$ for $\mathcal{M}$ and $\mathcal{N}$, we obtain 8 cardinal invariants of the continuum. Adding $\mathfrak{b}$ and $\mathfrak{d}$ to these invariants, we obtain 10 of them. The following fact tells us the relationship of these invariants.

Fact 2.0.2 (Bartoszyński, Fremlin, Miller, Rothberger and Truss (see [BJ95])). In the following diagram, an arrow drawn from a cardinal A to another cardinal B indicates that $A \leq B$ is provable from ZFC.

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[r] \arrow[u] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

This diagram is called Cichoń's diagram.

Let IP be the set of all interval partitions of ω. For $\bar{I} = \langle I_n : n \in \omega \rangle, \bar{J} = \langle J_m : m \in \omega \rangle \in \mathsf{IP}$, we define

$$\bar{I} < \bar{J} :\Leftrightarrow (\forall^\infty m)(\exists n)(I_n \subseteq J_m).$$

The following notion of Tukey relation is essential in the field of cardinal invariants.

Definition 2.0.3. (1) A triple $\mathbf{R} = (X, Y, R)$ is a relational system if $R \subseteq X \times Y$.

(2) For two relational system $\mathbf{R} = (X, Y, R)$ and $\mathbf{R}' = (X', Y', R')$, $\mathbf{R}$ is Tukey below $\mathbf{R}'$ if there are two maps $\varphi \colon X \to X'$ and $\psi \colon Y' \to Y$ such that $\varphi(x) R' y'$ implies $x R \psi(y')$ for every $x \in X$ and $y' \in Y'$.

(3) For a relational system $\mathbf{R} = (X, Y, R)$, we define

$$\mathfrak{d}(\mathbf{R}) = \min\{|B| : B \subseteq Y, (\forall x \in X)(\exists y \in B)(xRy)\}, \text{and}$$
$$\mathfrak{b}(\mathbf{R}) = \min\{|A| : A \subseteq X, (\forall y \in Y)(\exists x \in A)\neg(xRy)\}.$$

Definition 2.0.4. (1) For an ideal $\mathcal{I}$ on X, define $\mathbf{Cov}(\mathcal{I}) = (X, \mathcal{I}, \in)$.

(2) For an ideal $\mathcal{I}$ on X, define $\mathbf{Cof}(\mathcal{I}) = (\mathcal{I}, \mathcal{I}, \subseteq)$.

(3) $\mathbf{B} = (\omega^\omega, \omega^\omega, <^\infty)$.

(4) $\mathbf{B}^{\mathrm{IP}} = (\mathsf{IP}, \mathsf{IP}, \not>)$.

It is easy to see that $\mathrm{add}(\mathcal{I}) = \mathfrak{b}(\mathbf{Cof}(\mathcal{I}))$, $\mathrm{cof}(\mathcal{I}) = \mathfrak{d}(\mathbf{Cof}(\mathcal{I}))$, $\mathrm{non}(\mathcal{I}) = \mathfrak{b}(\mathbf{Cov}(\mathcal{I}))$, $\mathrm{cov}(\mathcal{I}) = \mathfrak{d}(\mathbf{Cov}(\mathcal{I}))$ and $\mathfrak{b}(\mathbf{B}) = \mathfrak{d}$ and $\mathfrak{d}(\mathbf{B}) = \mathfrak{b}$. It is well-known that $\mathbf{B}$ and $\mathbf{B}^{\mathrm{IP}}$ are Tukey equivalent (for example, see [Bla10, Theorem 2.10]).

An important fact on the Tukey relation is the following.

Fact 2.0.5. For two relational system $\mathbf{R}$ and $\mathbf{R}'$, if $\mathbf{R}$ is Tukey below $\mathbf{R}'$, then $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$ and $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ hold.

Definition 2.0.6. (1) For $c \in (\omega+1)^\omega, h \in \omega^\omega$, define $\prod c = \prod_{n\in\omega} c(n)$ and $S(c,h) = \prod_{n\in\omega}[c(n)]^{\leq h(n)}$.

(2) For $x \in \prod c$ and $\varphi \in S(c,h)$, define $x \in^* \varphi$ iff $(\forall^\infty n)(x(n) \in \varphi(n))$ and define $x \in^\infty \varphi$ iff $(\exists^\infty n)(x(n) \in \varphi(n))$.

We define cardinal invariants $\mathfrak{c}^\forall_{c,h}$ and $\mathfrak{v}^\forall_{c,h}$, which are called localization cardinals, and $\mathfrak{c}^\exists_{c,h}$ and $\mathfrak{v}^\exists_{c,h}$, which are called anti-localization cardinals.

Definition 2.0.7. (1) For $c \in (\omega+1)^\omega, h \in \omega^\omega$, define $\mathbf{Lc}(c,h) = (\prod c, S(c,h), \in^*)$, $\mathfrak{c}^\forall_{c,h} = \mathfrak{d}(\mathbf{Lc}(c,h))$ and $\mathfrak{v}^\forall_{c,h} = \mathfrak{b}(\mathbf{Lc}(c,h))$.

(2) Define $\mathbf{wLc}(c,h) = (\prod c, S(c,h), \in^\infty)$, $\mathfrak{c}^\exists_{c,h} = \mathfrak{d}(\mathbf{wLc}(c,h))$ and $\mathfrak{v}^\exists_{c,h} = \mathfrak{b}(\mathbf{wLc}(c,h))$.

Definition 2.0.8. (1) Define $\mathfrak{v}^\forall = \min\{\mathfrak{v}^\forall_{c,h} : c,h \in \omega^\omega, \lim_{n\to\infty} h(n) = \infty\}$.

(2) Define $\mathfrak{c}^\exists = \min\{\mathfrak{c}^\exists_{c,h} : c,h \in \omega^\omega, \sum_{n\in\omega} h(n)/c(n) < \infty\}$.

We also use the following higher cardinal invariants.

Definition 2.0.9. Let κ be a regular cardinal.

(1) For $x, y \in \kappa^\kappa$, y dominates x if there is $\alpha < \kappa$ such that for every $\beta \in [\alpha, \kappa)$ we have $x(\beta) < y(\beta)$.

(2) $\mathcal{A} \subseteq \kappa^\kappa$ is a dominating family if for every $x \in \kappa^\kappa$, there is $y \in \mathcal{A}$ that dominates x. Let $\mathfrak{d}_\kappa = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \kappa^\kappa$ a dominating family$\}$.

(3) $\mathcal{A} \subseteq \kappa^\kappa$ is an unbounded family if for every $x \in \kappa^\kappa$, there is $y \in \mathcal{A}$ that is not dominated by x. Let $\mathfrak{b}_\kappa = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \kappa^\kappa$ an unbounded family$\}$.

(4) We induce the topology on the set 2^κ using $<\kappa$-box topology. $\mathcal{M}_\kappa$ denotes the ideal of the all κ-unions of nowhere dense sets of 2^κ. Since $\mathcal{M}_\kappa$ is an ideal on 2^κ, we can use the notion $\mathrm{cov}(\mathcal{M}_\kappa)$ etc.

Here let us recall briefly the terminology in descriptive set theory. A pointclass is a class of subsets of Polish spaces. Examples are the class of all Borel subsets Borel, the class of all analytic sets $\mathbf{\Sigma}^1_1$, the class of all coanalytic sets $\mathbf{\Pi}^1_1$ and the class of all subsets all. Recall the pointclasses in projective hierarchy $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$: $\mathbf{\Sigma}^1_n$ is the class of all sets obtained by projection of $\mathbf{\Pi}^1_{n-1}$ sets along ω^ω and $\mathbf{\Pi}^1_n$ is the class of all sets whose complement is $\mathbf{\Sigma}^1_n$. Moreover we define $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$. We sometimes use the lightface version, such as Σ^1_n and Π^1_n, of pointclasses of the projective hierarchy. For more information on descriptive set theory, see [Mos09].

We require basic knowledge of forcing. We use the following well-known forcing notions.

Definition 2.0.10. (1) $\mathbb{C} = (2^{<\omega}, \supseteq)$, the Cohen forcing, which is forcing equivalent to $\mathsf{Borel}(2^\omega)/\mathcal{M}$.

(2) $\mathbb{B} = \mathsf{Borel}(2^\omega)/\mathcal{N}$, the random forcing.

(3) The Laver forcing $\mathbb{L}$. The conditions are all perfect subtrees $T \subseteq \omega^{<\omega}$ such that all nodes $\geq \mathrm{stem}(T)$ have infinitely many children. The order in $\mathbb{L}$ is $T' \leq T$ iff $T' \subseteq T$.

(4) For an ideal $\mathcal{I}$ of a Polish space X, let $\mathbb{P}_{\mathcal{I}} = \mathsf{Borel}(X)/\mathcal{I}$. This is called the idealized forcing of $\mathcal{I}$.

(5) For an ordinal κ, let $\mathrm{Coll}(\kappa)$ be the poset whose conditions are p such that p are finite partial functions and $\mathrm{dom}(p) \subseteq \kappa \times \omega$ and for every $(\alpha, n) \in \mathrm{dom}(p)$ we have $p(\alpha, n) \in \alpha$. The order is $q \leq p$ iff $p \subseteq q$. This is the Levy collapse.

Here we review basic notions of model theory.

Definition 2.0.11. (1) For a language $\mathcal{L}$, two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$ for every closed $\mathcal{L}$-formula φ.

(2) For a language $\mathcal{L}$, a sequence $\langle \mathcal{A}_i : i \in I \rangle$ of $\mathcal{L}$-structures and an ultrafilter $\mathcal{U}$ on I, we define their ultraproduct $\prod_{i \in I} \mathcal{A}/\mathcal{U}$ taking the quotient of the product set $\prod_{i \in I} \mathcal{A}$ by the equivalence relation $x \sim y \iff \{i \in I : x(i) = y(i)\} \in \mathcal{U}$. Evaluations of the symbols in $\mathcal{L}$ are defined naturally. When all $\mathcal{A}_I$ are equal to the same structure $\mathcal{A}$, the ultraproduct is called ultrapower and the symbol $\mathcal{A}^{\mathcal{U}}$ denotes it.

(3) Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure. A set $p(x)$ of $\mathcal{L}(\mathcal{A})$-formulas with one variable x is finitely satisfiable if every finite subset of $p(x)$ has a solution in $\mathcal{A}$. $\mathcal{A}$ is saturated if every set $p(x)$ of $\mathcal{L}(\mathcal{A})$-formulas with one variable x which is finitely satisfiable and the elements in $\mathcal{A}$ occurring in $p(x)$ is of size $<|\mathcal{A}|$ has a solution in $\mathcal{A}$.

We review basic notions of ultrafilters.

Definition 2.0.12. (1) An ultrafilter $\mathcal{U}$ on a set I is *uniform* if $|A| = |I|$ for every $A \in \mathcal{U}$.

(2) Let $\mathcal{U}$ be a ultrafilter on κ. We say $\mathcal{U}$ is *regular* if there is $\mathcal{E} \subseteq \mathcal{U}$ of size κ such that for every $i < \kappa$, the set $\{E \in \mathcal{E} : i \in E\}$ is finite.

(3) For ultrafilters $\mathcal{U}, \mathcal{V}$ on I, J respectively, we define

$$\mathcal{U} * \mathcal{V} = \{A \subseteq I \times J : \{i \in I : \{j \in J : (i, j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

$\mathcal{U} * \mathcal{V}$ is called the *Fubini product* of $\mathcal{U}$ and $\mathcal{V}$.

(4) An ultrafilter $\mathcal{U}$ on a set I is *good* if for every $f \colon [I]^{<\omega} \to \mathcal{U}$ satisfying the condition $a \subseteq b$ implies $f(a) \supseteq f(b)$, there is a $g \colon [I]^{<\omega} \to \mathcal{U}$ such that for every $a, b \in [I]^{<\omega}$ we have $g(a) \subseteq f(a)$ and $g(a \cup b) = g(a) \cap g(b)$.

(5) Let βI be the set of all ultrafilters on I and let $\beta I \smallsetminus I$ be the set of all non-principal ultrafilters on I.

(6) For ultrafilters $\mathcal{U}, \mathcal{V}$ on I, J, respectively, $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ if there is $f \colon I \to J$ such that

$$\mathcal{V} = \{Y \subseteq J : f^{-1}(Y) \in \mathcal{U}\}.$$

This order is called Rudin–Keisler ordering.

We now recall basic definitions from determinacy. Basic information about it can be found in [Mos09, Chapter 6].

Definition 2.0.13. Let $A \subseteq \omega^\omega$. Consider the following game associated with A: Player I and II play in turn natural numbers.

$$\begin{array}{l|cccccc} \text{Player I} & n_0 & & n_2 & & \dots & \\ \hline \text{Player II} & & n_1 & & n_3 & & \dots \end{array}$$

Player I wins if $\langle n_0, n_1, n_2, \dots \rangle \in A$.

We say A is determined if either player has a winning strategy.

For a pointclass Γ, $\mathrm{Det}(\Gamma)$ is the statement that A is determined for every $A \in \Gamma$. This is the axiom of determinacy for Γ. AD stands for the full axiom of determinacy, that is $\mathrm{Det}(\mathsf{all})$.

Chapter 3

Goldstern's principle

In [Gol93], Goldstern showed the following theorem: let $\langle A_x : x \in \omega^\omega \rangle$ be a family of Lebesgue measure zero sets. Assume that this family is monotone in the sense that if $x, x' \in \omega^\omega$ satisfies $x \leq x'$ then $A_x \subseteq A_{x'}$. Also assume that $A = \{(x, y) : y \in A_x\}$ is a $\boldsymbol{\Sigma}^1_1$ set. Then $\bigcup_{x \in \omega^\omega} A_x$ has also Lebesgue measure zero. Goldstern stated this theorem in terms of complements and applied it to uniform distribution theory. Our main focus is to study to what extent we can remove this $\boldsymbol{\Sigma}^1_1$ assumption.

3.1 Review of Goldstern's proof

In [Gol93], Goldstern proved the following theorem. In the proof, he uses the Shoenfield absoluteness theorem and the random forcing. As for these, see [Kan08, Chapter 3].

Theorem 3.1.1 (Goldstern)**.** Let (Y, μ) be a Polish probability space. Let $A \subseteq \omega^\omega \times Y$ be a $\boldsymbol{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has measure 0. Also, assume that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ also has measure 0.

Proof. We may assume that $Y = 2^\omega$ and μ is the usual measure of 2^ω since for every Polish probability space Y, there is a Borel isomorphism between measure 1 subsets of Y and 2^ω that preserves measure.

Fix a defining formula and a parameter of A. In generic extensions if we write A, we refer to the set defined by the formula and the parameter in the model.

Since A and $\bigcup_{x \in \omega^\omega} A_x$ are $\boldsymbol{\Sigma}^1_1$ sets, they are Lebesgue measurable. Toward a contradiction, assume that $B := \bigcup_{x \in \omega^\omega} A_x$ does not have measure 0. Then B has positive measure. By inner regularity of the measure, we can take a closed set $K \subseteq B$ with positive measure. Take a Borel code k of K. We take a random real $r \in 2^\omega$ over V such that $r \in \hat{k}$.

Now for each $x \in \omega^\omega \cap V$, we have $r \notin A_x$. In order to prove it, take a Borel code d_x such that $A_x \subseteq \hat{d_x}$ and $\mu(\hat{d_x}) = 0$. But the condition $A_x \subseteq \hat{d_x}$ is $\boldsymbol{\Pi}^1_1$. Thus, since the random real avoids $\hat{d_x}$, we have $r \notin A_x$.

Therefore we have

$$r \notin \bigcup_{x \in \omega^\omega \cap V} A_x.$$

But in $V[r]$, it also holds that

$$(\forall x, x' \in \omega^\omega)(x \leq x' \to A_x \subseteq A_{x'})$$

since this formula is $\mathbf{\Pi}^1_2$. Thus, by the assumption that A_x is increasing and the fact that the random forcing is ω^ω-bounding, this implies

$$r \notin \bigcup_{x \in \omega^\omega} A_x.$$

Therefore, in $V[r]$, it holds that

$$(\exists r' \in 2^\omega)(r' \in \hat{k} \setminus B)$$

because $r' = r$ suffices. Recalling B is a $\mathbf{\Sigma}^1_1$ set, this statement is written by a $\mathbf{\Sigma}^1_2$ formula. Therefore, by Shoenfield's absoluteness, it holds also in V. That is, there exists an $r' \in 2^\omega$ in V such that

$$r' \in K \setminus B.$$

This contradicts the choice of K. $\square$

We define the principle $\mathsf{GP}(\Gamma)$. We call the condition $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$ the monotonicity condition for A.

Definition 3.1.2. Let Γ be a pointclass. Then $\mathsf{GP}(\Gamma)$ means the following statement: Let (Y, μ) be a Polish probability space and $A \subseteq \omega^\omega \times Y$ be in Γ. Assume the monotonicity condition for A. Also suppose that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also μ-measure 0.

We define $\mathsf{GP}^*(\Gamma)$ as $\mathsf{GP}(\Gamma)$ by replacing $\leq$ by $\leq^*$.

By Goldstern's theorem, we have $\mathsf{GP}(\mathbf{\Sigma}^1_1)$.

For the reasons stated in the proof of Theorem 3.1.1, if the pointclass Γ contains all Borel sets and closed under Borel functions, then we may assume that the space (Y, μ) in the definition of $\mathsf{GP}(\Gamma)$ is the Cantor space.

Theorem 3.1.3. For every natural number n, if $\mathbf{\Sigma}^1_{n+1}$-$\mathbb{B}$-absoluteness holds and every $\mathbf{\Sigma}^1_n$ set is Lebesgue measurable, then $\mathsf{GP}(\mathbf{\Sigma}^1_n)$ holds. In particular, if every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable, then $\mathsf{GP}(\mathbf{\Sigma}^1_2)$ holds.

Proof. This is proved by the same argument as Theorem 3.1.1. Recall that $\mathbf{\Sigma}^1_3$-$\mathbb{B}$-absoluteness follows from $\mathbf{\Sigma}^1_2$ measurability (see [BJ95, Theorem 9.2.12 and 9.3.8]). $\square$

Clearly $\mathsf{GP}(\Gamma)$ implies $\mathsf{GP}^*(\Gamma)$. If we make a slight assumption on the pointclass Γ, then the converse holds. We only consider such pointclasses.

Lemma 3.1.4. If a pointclass Γ is closed under recursive substitution and projection along ω, then $\mathsf{GP}^*(\Gamma) \Rightarrow \mathsf{GP}(\Gamma)$.

Proof. Assume that $A \in \Gamma$ and for each $x \in \omega^\omega$, A_x has μ-measure 0 and that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Put $B_x = \bigcup\{A_y : x \text{ and } y \text{ are almost equal}\}$.

Then by assumption, $B \in \Gamma$ and for each $x \in \omega^\omega$, B_x has μ-measure 0 and that $(\forall x, x' \in \omega^\omega)(x \leq^* x' \Rightarrow B_x \subseteq B_{x'})$. Therefore, by $\mathsf{GP}^*(\Gamma)$, $\bigcup_{x \in \omega^\omega} B_x$ is a measure 0 set. Thus $\bigcup_{x \in \omega^\omega} A_x$ is a measure 0 set. $\square$

3.2 $\mathsf{GP}(\mathbf{\Pi}^1_1)$

In this section, we prove that $\mathsf{GP}(\mathbf{\Pi}^1_1)$ holds.

Fact 3.2.1 ([Kec73; Tan67])**.** Let $U \in \Sigma^1_1$, $U \subseteq \omega^\omega \times 2^\omega$ be the universal for $\mathbf{\Sigma}^1_1$ subset of 2^ω. Then the relation $\mu(U_x) > r$ for $x \in \omega^\omega$ and $r \in \mathbb{R}$ is Σ^1_1.

Corollary 3.2.2. Let $A \subseteq \omega^\omega \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Then the relation $\mu(A_x) > r$ for $x \in \omega^\omega$ and $r \in \mathbb{R}$ is $\mathbf{\Sigma}^1_1$.

Proof. Take universal sets U and $U^{(2)}$ for $\mathbf{\Sigma}^1_1$ subsets of 2^ω and $\omega^\omega \times 2^\omega$, respectively, with the following coherent property: $U(S(e,x),y) \iff U^{(2)}(e,x,y)$, where S is a recursively continuous function. As for existence of such coherent universal sets, see [Mos09, Section 3.H]. Take $e \in \omega^\omega$ such that $A(x,y) \iff U^{(2)}(e,x,y)$. Then we have

$$\mu(A_x) > r \iff \mu(U_{S(e,x)}) > r,$$

which is a $\mathbf{\Sigma}^1_1$ relation. $\square$

Corollary 3.2.3. Let $A \subseteq \omega^\omega \times 2^\omega$ be a $\mathbf{\Pi}^1_1$ set. Then the relation $\mu(A_x) = 0$ for $x \in \omega^\omega$ is $\mathbf{\Sigma}^1_1$.

Proof. Let $B = (\omega^\omega \times 2^\omega) \smallsetminus A$, which is $\mathbf{\Sigma}^1_1$ set. We have

$$\mu(A_x) = 0 \iff \mu(B_x) = 1 \iff (\forall n)(\mu(B_x) > 1 - 1/2^n),$$

which is a $\mathbf{\Sigma}^1_1$ relation. $\square$

Theorem 3.2.4. $\mathsf{GP}(\mathbf{\Pi}^1_1)$ holds.

Proof. Let $A \subseteq \omega^\omega \times 2^\omega$ be a $\mathbf{\Pi}^1_1$ set. Assume $\langle A_x : x \in \omega^\omega \rangle$ is monotone and each A_x is null. Take a Laver real d over V. $(\forall x \in \omega^\omega)(\mu(A_x) = 0)$ holds in V and this sentence is $\mathbf{\Pi}^1_2$ using Corollary 3.2.3. So in $V[d]$, $\mu(A_d) = 0$ holds. Also monotonicity of $\langle A_x : x \in \omega^\omega \rangle$ can be written as a $\mathbf{\Pi}^1_2$ formula and holds in V, so it holds also in $V[d]$. Since d is a dominating real over V, we have $(\bigcup_{x \in \omega^\omega} A_x)^V \subseteq \bigcup_{x \in \omega^\omega \cap V} A_x \subseteq A_d$. Therefore $(\bigcup_{x \in \omega^\omega} A_x)^V$ is null in $V[d]$. Since Laver forcing preserves Lebesgue outer measure, it holds that $\bigcup_{x \in \omega^\omega} A_x$ is null in V. $\square$

3.3 Consistency of $\neg\mathsf{GP}(\mathsf{all})$

In this section, we assume ZFC.

Definition 3.3.1. We call a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ a *null tower* if it is an increasing sequence of measure 0 sets such that its union does not have measure 0.

Theorem 3.3.2. If there is a null tower of length either $\mathfrak{b}$ or $\mathfrak{d}$, then $\neg\mathsf{GP}(\mathsf{all})$ holds.

Proof. In the case of $\mathfrak{b}$: By assumption, we take an increasing sequence $\langle A_\alpha : \alpha < \mathfrak{b} \rangle$ of measure-0 sets such that $\bigcup_{\alpha < \mathfrak{b}} A_\alpha$ doesn't have measure 0. We can take an increasing and unbounded sequence $\langle x_\alpha : \alpha < \mathfrak{b} \rangle$ with respect to $\leq^*$. (This sequence is not necessarily cofinal.) For each $x \in \omega^\omega$, put

$$\alpha(x) = \min\{\alpha < \mathfrak{b} : x_\alpha \not\leq^* x\}.$$

This is well-defined since $\langle x_\alpha : \alpha < \mathfrak{b} \rangle$ is unbounded. And then put

$$B_x = A_{\alpha(x)}.$$

Now each B_x has measure 0 and we have

$$\begin{aligned} x \leq x' &\Rightarrow x \leq^* x' \\ &\Rightarrow (\forall \alpha)(x_\alpha \leq^* x \Rightarrow x_\alpha \leq^* x') \\ &\Rightarrow \{\alpha : x_\alpha \not\leq^* x'\} \subseteq \{\alpha : x_\alpha \not\leq^* x\} \\ &\Rightarrow \alpha(x) \leq \alpha(x') \\ &\Rightarrow B_x \subseteq B_{x'}. \end{aligned}$$

Thus $\langle B_x : x \in \omega^\omega \rangle$ is increasing. Also we have $\bigcup_{x \in \omega^\omega} B_x = \bigcup_{\alpha < \mathfrak{b}} A_\alpha$. Indeed, it is obvious that the left-hand side is contained in the right-hand side. To prove the reverse inclusion, it is sufficient to each A_α is contained in some B_x. So fix α and consider $x = x_\alpha$. Since the sequence $\langle x_\alpha : \alpha < \mathfrak{b} \rangle$ is increasing, we have $\alpha \leq \alpha(x)$. Thus $A_\alpha \subseteq A_{\alpha(x)} = B_x$.

Therefore, $\bigcup_{x \in \omega^\omega} B_x$ doesn't have measure 0.

In the case of $\mathfrak{d}$: As above, we can take an increasing sequence $\langle A_\alpha : \alpha < \mathfrak{d} \rangle$ of measure-0 sets such that $\bigcup_{\alpha < \mathfrak{d}} A_\alpha$ doesn't have measure 0. By the definition of $\mathfrak{d}$, we can take a dominating sequence $\langle x_\alpha : \alpha < \mathfrak{d} \rangle$ with respect to $\leq^*$. (This sequence is not necessarily increasing.)

For each $x \in \omega^\omega$, put

$$\alpha(x) = \min\{\alpha < \mathfrak{d} : x \leq^* x_\alpha\}$$

and put

$$B_x = A_{\alpha(x)}.$$

One can easily show that $\langle B_x : x \in \omega^\omega \rangle$ is increasing. Also we have $\bigcup_{x \in \omega^\omega} B_x = \bigcup_{\alpha < \mathfrak{d}} A_\alpha$. That the left-hand side is contained in the right-hand side is obvious. To show the reverse inclusion, fix α. Since the sequence $\langle x_\beta : \beta < \alpha \rangle$ is not a dominating family, we can find an $x \in \omega^\omega$ such that for all $\beta < \alpha$, $x \not\leq^* x_\beta$. Then $\alpha \leq \alpha(x)$. Thus we have $A_\alpha \subseteq A_{\alpha(x)} = B_x$. □

Corollary 3.3.3. Assume that at least one of the following three conditions holds:

(1) $\mathrm{add}(\mathcal{N}) = \mathfrak{b}$,

(2) $\mathrm{non}(\mathcal{N}) = \mathfrak{b}$ or

(3) $\mathrm{non}(\mathcal{N}) = \mathfrak{d}$.

Then $\neg\mathsf{GP}(\mathsf{all})$ holds. In particular the continuum hypothesis implies $\neg\mathsf{GP}(\mathsf{all})$.

Proof. Clearly there are null towers of length both $\mathrm{add}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$. So using Theorem 3.3.2, we have this corollary. □

Proposition 3.3.4. $\mathsf{GP}(\mathsf{all})$ implies $\mathrm{add}(\mathcal{M}) < \mathrm{cof}(\mathcal{M})$.

Proof. Assume $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$. Let $\langle M_\alpha : \alpha < \kappa \rangle$ be a cofinal increasing sequence of meager sets. We can take such a sequence since $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \kappa$. For each $\alpha < \kappa$, take $x_\alpha \in M_{\alpha+1} \setminus M_\alpha$.

Now recall from Rothberger's theorem, there is a Tukey morphism $(\varphi, \psi) \colon (2^\omega, \mathcal{N}, \in) \to (\mathcal{M}, 2^\omega, \not\ni)$. That is, there are $\varphi \colon 2^\omega \to \mathcal{M}$ and $\psi \colon 2^\omega \to \mathcal{N}$ such that $\varphi(x) \not\ni y$ implies $x \in \psi(y)$ for every $x, y \in 2^\omega$.

Using this theorem, we put $N_\alpha = \bigcap_{\beta \geq \alpha} \psi(x_\beta)$ for $\alpha < \kappa$. Then $\langle N_\alpha : \alpha < \kappa \rangle$ is a sequence of null sets of length $\kappa = \mathfrak{b}$ and its union is 2^ω. □

In the following proposition, we show that $\mathsf{GP}(\mathsf{all})$ cannot be forced by finite support iteration of ccc forcings.

Proposition 3.3.5. For every finite support iteration of ccc forcings $\langle P_\alpha : \alpha < \nu \rangle$ with $\mathrm{cf}(\nu) \geq \aleph_1$, we have $P_\nu \Vdash \neg\mathsf{GP}(\mathsf{all})$.

Proof. Let G be a (V, P_ν) generic filter and work in $V[G]$. Let $\langle c_\alpha : \alpha < \mathrm{cf}(\nu) \rangle$ be a sequence of Cohen reals added cofinally by P_α.

For a Cohen real c, let $\mathrm{nullset}(c)$ denote the standard null set constructed from c.

We have the following:

- For every $x \in \omega^\omega$, there is $\alpha < \mathrm{cf}(\nu)$ such that for every $\beta > \alpha$ we have $c_\beta \not\leq^* x$.
- For every $z \in 2^\omega$, there is $\alpha < \mathrm{cf}(\nu)$ such that for every $\beta > \alpha$ we have $z \in \mathrm{nullset}(c_\beta)$.

For $x \in \omega^\omega$, we let $\alpha_x = \min\{\alpha : (\forall \beta > \alpha)(c_\beta \not\leq^* x)\}$ and let $A_x = \bigcap_{\beta > \alpha_x} \mathrm{nullset}(c_\beta)$.

We can easily show that each A_x is a null set, the sequence $\langle A_x : x \in \omega^\omega \rangle$ is increasing and the union $\bigcup_{x \in \omega^\omega} A_x$ is equal to 2^ω. Therefore, $\langle A_x : x \in \omega^\omega \rangle$ is a witness of $\neg\mathsf{GP}(\mathsf{all})$. □

3.4 Consistency of $\mathsf{GP}(\mathsf{all})$

In this section, as in the previous section, we assume ZFC. To obtain a model of $\mathsf{GP}(\mathsf{all})$, $\mathrm{add}(\mathcal{N}) \neq \mathfrak{b}$, $\mathrm{non}(\mathcal{N}) \neq \mathfrak{b}$, $\mathrm{non}(\mathcal{N}) \neq \mathfrak{d}$ and $\mathrm{add}(\mathcal{M}) \neq \mathrm{cof}(\mathcal{M})$ need to hold. A natural model in which they hold is the Laver model. In this section, we will see that $\mathsf{GP}(\mathsf{all})$ actually holds in the Laver model.

Theorem 3.4.1. Assume that $\mathfrak{b} = \mathfrak{d}$ and let both of these be κ. Then the following are equivalent.

(1) There is a null tower of length κ.

(2) $\neg\mathsf{GP}(\mathsf{all})$.

Proof. That (1) implies (2) is shown in Theorem 3.3.2.

We now prove that (2) implies (1). Assume that $\neg\mathsf{GP}^*(\mathsf{all})$. Then we can take $A \subseteq \omega^\omega \times 2^\omega$ such that each section A_x has measure 0 and $(\forall x, x' \in \omega^\omega)(x \leq^* x' \Rightarrow A_x \subseteq A_{x'})$ holds and $B = \bigcup_{x \in \omega^\omega} A_x$ does not have measure 0. By $\mathfrak{b} = \mathfrak{d} = \kappa$, we can take a cofinal increasing sequence $\langle x_\alpha : \alpha < \kappa \rangle$ with respect to $\leq^*$. For each $\alpha < \kappa$, put $C_\alpha = A_{x_\alpha}$. Then each C_α has measure 0. Since $\alpha \mapsto x_\alpha$ is increasing and $x \mapsto A_x$ is increasing, $\langle C_\alpha : \alpha < \kappa \rangle$ is also increasing. Also, since $\langle x_\alpha : \alpha < \kappa \rangle$ is cofinal, we have $B = \bigcup_{\alpha < \kappa} C_\alpha$. So $\bigcup_{\alpha < \kappa} C_\alpha$ does not have measure 0. Thus $\langle C_\alpha : \alpha < \kappa \rangle$ is a null tower of length κ. □

The following lemma and theorem requires knowledge of proper forcing. See [Gol92].

Lemma 3.4.2. Assume CH. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration of proper forcing notions such that

$$\Vdash_\alpha |\dot{Q}_\alpha| \leq \mathfrak{c} \quad (\text{for all } \alpha < \omega_2).$$

Let $\langle \dot{X}_\alpha : \alpha < \omega_2 \rangle$ be a sequence of P_{ω_2}-names such that

$$\Vdash_{\omega_2} (\forall \alpha < \omega_2)(\dot{X}_\alpha \text{ has measure } 0).$$

Then the set

$$C = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1 \;\& \\ \Vdash_{\omega_2} (\langle \dot{X}_\beta \cap V[\dot{G}_\alpha] : \beta < \alpha\rangle \in V[\dot{G}_\alpha] \;\& \; (\forall \beta < \alpha)(\dot{X}_\beta \cap V[\dot{G}_\alpha] \text{ has measure } 0)^{V[\dot{G}_\alpha]})\}.$$

contains a ω_1-club set in ω_2.

Proof. This is an example of a reflection argument. See also [Hal12, Chapter 26].

Take a sequence $\langle \dot{c}_\beta : \beta < \omega_2\rangle$ of names of Borel codes such that

$$\Vdash_{\omega_2} (\forall \beta < \omega_2)(\dot{X}_\beta \subseteq \hat{\dot{c}}_\beta \;\&\; \hat{\dot{c}}_\beta \text{ has measure } 0).$$

For each $\beta < \omega_2$, take $\gamma_\beta < \omega_2$ such that $\dot{c}_\beta$ is a P_{γ_β}-name.

Since for each $\alpha < \omega_2$, $\Vdash_\alpha$ CH, we can take a sequence $\langle \dot{x}_i^\alpha : i < \omega_1\rangle$ such that

$$\Vdash_\alpha \text{“}\langle \dot{x}_i^\alpha : i < \omega_1\rangle \text{ is an enumeration of } 2^\omega\text{”}$$

. For each $\alpha, \beta < \omega_2$ and $i < \omega_1$, take a maximal antichain $A_i^{\alpha,\beta}$ such that

$$A_i^{\alpha,\beta} \subseteq \{p \in P_{\omega_2} : p \Vdash \dot{x}_i^\alpha \in \dot{X}_\beta \text{ or } p \Vdash \dot{x}_i^\alpha \notin \dot{X}_\beta\}.$$

Since P_{ω_2} has ω_2-cc, we can take $\delta_i^{\alpha,\beta} < \omega_2$ such that

$$\bigcup \{\mathrm{supt}(p) : p \in A_i^{\alpha,\beta}\} \subseteq \delta_i^{\alpha,\beta}.$$

We define a function f from ω_2 into ω_2 as follows:

$$f(\nu) = \sup\left(\{\gamma_\beta : \beta \leq \nu\} \cup \{\delta_i^{\alpha,\beta} : \alpha, \beta \leq \nu, i < \omega_1\}\right)$$

Put

$$C' = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1, (\forall \nu < \alpha) f(\nu) < \alpha\}.$$

Then clearly C' is ω_1-club set. So it suffices to show that $C' \subseteq C$.

Let $\alpha \in C'$ and we shall prove $\alpha \in C$. Fix $\beta < \alpha$. Define a P_α-name $\dot{Y}$ by

$$\Vdash_\alpha \text{“}\dot{Y} = \bigcup_{\alpha' < \alpha} \{\dot{x}_i^{\alpha'} : (p \Vdash \dot{x}_i^{\alpha'} \in \dot{X}_\beta)^V \\ \text{for some } p \in A_i^{\alpha',\beta} \;\&\; p \restriction \alpha \in \dot{G}\}\text{”}. \tag{$*$}$$

We claim that $\Vdash_{\omega_2} \dot{X}_\beta \cap V[\dot{G}_\alpha] = \dot{Y}$. In order to prove this, take a (V, P_{ω_2})-generic filter G. In $V[G]$, take $x \in \dot{X}_\beta^G \cap V[G_\alpha]$. Since no new real is added at stage α, we can take $\alpha' < \alpha$ such that $x \in V[G_{\alpha'}]$. Thus there is $i < \omega_1$ such that $x = (\dot{x}_i^{\alpha'})^G$. Since $(\dot{x}_i^{\alpha'})^G \in \dot{X}_\beta^G$, in V, we can take a $p \in G \cap A_i^{\alpha',\beta}$ such that $p \Vdash \dot{x}_i^{\alpha'} \in \dot{X}_\beta$. We have $p \in A_i^{\alpha',\beta}$. Thus x is an element of $\dot{Y}^G$.

Conversely, take an element x of $\dot{Y}^G$. So we can take $\alpha' < \alpha$, $i < \omega_1$ and $p \in P_{\omega_2}$ such that

$$x = (\dot{x}_i^{\alpha'})^G, (p \Vdash \dot{x}_i^{\alpha'} \in \dot{X}_\beta)^V, p \in A_i^{\alpha',\beta} \;\&\; p \restriction \alpha \in G_\alpha.$$

Clearly we have $x \in V[G_{\alpha'}] \subseteq V[G_\alpha]$. Suppose that $(\dot{x}_i^{\alpha'})^G \notin \dot{X}_\beta^G$. Then we can take $q \in G$ such that $q \Vdash \dot{x}_i^{\alpha'} \notin \dot{X}_\beta$. By the maximality of $A_i^{\alpha',\beta}$, we can take $r \in A_i^{\alpha',\beta} \cap G$. Since both q and r are elements of G, q and r are compatible. So $r \Vdash \dot{x}_i^{\alpha'} \notin \dot{X}_\beta$. Thus p and r are incompatible. But $\operatorname{supt}(p), \operatorname{supt}(r) \subseteq \alpha$. So $p \restriction \alpha$ and $r \restriction \alpha$ are incompatible. But they are elements of G_α. It contradicts that G_α is a (V, P_α)-generic filter.

Thus we have $\Vdash_{\omega_2} \dot{X}_\beta \cap V[G_\alpha] \in V[G_\alpha]$.

By performing the above operations simultaneously with respect to the β, we have

$$\Vdash_{\omega_2} \langle \dot{X}_\beta \cap V[\dot{G}_\alpha] : \beta < \alpha \rangle \in V[\dot{G}_\alpha].$$

Since we have $\Vdash_{\omega_2} \dot{X}_\beta \subseteq \hat{c}_\beta$, it holds that

$$\Vdash_{\omega_2} \text{“}\dot{X}_\beta \cap V[G_\alpha] \subseteq \hat{c}_\beta \text{ has measure 0”}.$$

Therefore, we have $\alpha \in C$. □

Recall that $\mathbb{L}$ denotes the Laver forcing. As for basic properties of Laver forcing, see [BJ95].

Theorem 3.4.3. Assume CH. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ be the countable support iteration such that

$$\Vdash_\alpha \dot{Q}_\alpha = \mathbb{L} \text{ (for all } \alpha < \omega_2).$$

Then

$$\Vdash_{\omega_2} \mathsf{GP}(\mathsf{all}).$$

In particular, if ZFC is consistent then so is $\mathsf{ZFC} + \mathsf{GP}(\mathsf{all})$.

Proof. By Theorem 3.4.1 and the fact that $\Vdash_{\omega_2} \mathfrak{b} = \mathfrak{d} = \omega_2$, it is sufficient to show that

$$\Vdash_{\omega_2} \text{“There is no null tower of length } \omega_2\text{”}.$$

Let G be a (V, P_{ω_2})-generic filter. In $V[G]$, consider an increasing sequence $\langle A_\alpha : \alpha < \omega_2 \rangle$ of measure 0 sets. By Lemma 3.4.2, we can find a stationary set $S \subseteq \omega_2$ such that for all $\alpha \in S$, $\operatorname{cf}(\alpha) = \omega_1$ and

$$(\langle A_\beta \cap V[G_\alpha] : \beta < \alpha \rangle \in V[G_\alpha] \,\&\, (\forall \beta < \alpha)((A_\beta \cap V[G_\alpha] \text{ has measure } 0)^{V[G_\alpha]}).$$

Fix $\alpha \in S$. Put $B_\alpha := \bigcup_{\beta<\alpha} A_\beta \cap V[G_\alpha]$. Then we have $\bigcup_{\alpha<\omega_2} B_\alpha = \bigcup_{\alpha<\omega_2} A_\alpha$. We now prove that B_α is also a measure 0 set in $V[G_\alpha]$. Let α' be the successor of α in S. Then B_α is a measure 0 set in $V[G_{\alpha'}]$. Since the quotient forcing $P_{\alpha'}/G_\alpha$ is a countable suppport iteration of the Laver forcing, this forcing preserves positive outer measure. So B_α is also a measure 0 set in $V[G_\alpha]$.

For each $\alpha \in S$, take a Borel code $c_\alpha \in \omega^\omega$ of a measure 0 set such that $B_\alpha \subseteq \hat{c}_\alpha$ in $V[G_\alpha]$. Since $\operatorname{cf}(\alpha) = \omega_1$, each c_α appears a prior stage. Then by Fodor's lemma, we can take a stationary set $S' \subseteq \omega_2$ that is contained by S and $\beta < \omega_2$ such that $(\forall \alpha \in S')(c_\alpha \in V[G_\beta])$. But the number of reals in $V[G_\beta]$ is $\aleph_1$, so we can take $T \subseteq S'$ unbounded in ω_2 and c such that $(\forall \alpha \in T)(c_\alpha = c)$. Then we have $\bigcup_{\alpha<\omega_2} A_\alpha \subseteq \hat{c}$ in $V[G]$. So $\bigcup_{\alpha<\omega_2} A_\alpha$ has measure 0. □

Corollary 3.4.4. $\operatorname{Con}(\mathsf{ZFC}) \to \operatorname{Con}(\mathsf{ZFC} + \mathsf{GP}(\mathsf{projective}) + \neg\mathsf{GP}(\mathsf{all}))$. Here, $\mathsf{projective} = \bigcup_{n\geq 1} \mathbf{\Sigma}^1_n$.

Proof. Assume CH and let P be the forcing poset from Theorem 3.4.3, that is the countable support iteration of Laver forcing notions of length ω_2. Then we have $P \Vdash \mathsf{GP}(\mathsf{all})$. In particular we have

$P \Vdash \mathsf{GP}(\mathsf{projective})$. Let $\dot{Q}$ be a P-name of the poset

$$\mathrm{Fn}(\omega_1, 2, \omega_1) = \{p : p \text{ is a countable partial function from } \omega_1 \text{ to } 2\}$$

with the reverse inclusion order. It is well-known that provably $\mathrm{Fn}(\omega_1, 2, \omega_1)$ adds no new reals and forces CH. So we have $P * \dot{Q} \Vdash$ CH. Since $P \Vdash \mathsf{GP}(\mathsf{projective})$ and $P \Vdash$ "$\dot{Q}$ adds no new reals", we have $P * \dot{Q} \Vdash \mathsf{GP}(\mathsf{projective})$. □

3.5 Consequences of determinacy

In this section, we don't assume the axiom of choice and we will discuss a consequence of determinacy for Goldstern's principle.

Theorem 3.5.1. Let Γ be a pointclass that contains all Borel subsets and is closed under Borel substitution. Assume $\mathrm{Det}(\Gamma)$. Then $\mathsf{GP}(\mathrm{proj}(\Gamma))$ holds, where $\mathrm{proj}(\Gamma)$ is the pointclass of all projections along ω^ω of a set in Γ.

In particular, AD implies $\mathsf{GP}(\mathsf{all})$. Also $\mathrm{Det}(\boldsymbol{\Pi}^1_n)$ implies $\mathsf{GP}(\boldsymbol{\Sigma}^1_{n+1})$ for every $n \geq 1$.

Proof. This proof is based on Harrington's covering game. See also [Mos09, Exercise 6A.17]. In this proof, we use the following notation: for $j < n < \omega$,

$$\mathrm{proj}^n_j : (\omega^\omega)^n \to (\omega^\omega)^{n-1}; (x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}).$$

Fix $B \subseteq \omega^\omega \times \omega^\omega \times 2^\omega$ and $A = \mathrm{proj}^3_0(B)$ such that each section A_x has measure 0, $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Also let $\varepsilon > 0$. We have to show that the outer measure $\mu^*(\mathrm{proj}(A))$ is less than or equal to ε.

Fix a Borel isomorphism $\pi \colon 2^\omega \to \omega^\omega$. Consider the following game: At stage n, player I plays $(s_n, t_n, u_n) \in \{0,1\}^3$. Player II then plays a finite union G_n of basic open sets such that $\mu(G_n) \leq \varepsilon/16^{n+1}$. In this game, we define that player I wins if and only if $(z, x, y) \in B$ and $y \notin \bigcup_{n\in\omega} G_n$, where $x = \pi(s_0, s_1, \dots), y = (t_0, t_1, \dots)$ and $z = \pi(u_0, u_1, \dots)$.

$$\begin{array}{lcccccccc}
\text{Player I} & (s_0, t_0, u_0) & & (s_1, t_1, u_1) & & (s_2, t_2, u_2) & & & \\
 & & \searrow & \nearrow & \searrow & \nearrow & \searrow & & \cdots \\
\text{Player II} & & G_0 & & G_1 & & G_2 & &
\end{array}$$

Assume that player I has a winning strategy σ. Put

$$C = \{(z, x, y) \in \omega^\omega \times \omega^\omega \times 2^\omega : (\exists (G_0, G_1, \dots))((z, x, y) \text{ is the play of I along } \sigma \text{ against } (G_0, G_1, \dots))\}.$$

Then clearly C is a $\boldsymbol{\Sigma}^1_1$ set. Since player I wins, we have $C \subseteq B$. So we have $\mathrm{proj}^3_0(C) \subseteq \mathrm{proj}^3_0(B) = A$. So each $(\mathrm{proj}^3_0(C))_x \subseteq A_x$ has measure 0. For $x \in \omega^\omega$, put $D_x = \bigcup_{x' \leq x} (\mathrm{proj}^3_0(C))_{x'}$, which is a $\boldsymbol{\Sigma}^1_1$ set. Since $(\mathrm{proj}^3_0(C))_x \subseteq A_x$, each D_x has measure 0. And we have $x' \leq x$ implies $D_{x'} \subseteq D_x$.

Thus, by $\mathsf{GP}(\boldsymbol{\Sigma}^1_1)$, $\mathrm{proj}^2_0(D)$ has measure 0. So $\mathrm{proj}^2_0(\mathrm{proj}^3_0(C))$ has also measure 0. Therefore we can take $(G_0, G_1, \dots)$ such that $\mathrm{proj}^2_0(\mathrm{proj}^3_0(C)) \subseteq \bigcup_{n\in\omega} G_n$ and $\mu(G_n) \leq \varepsilon/16^{n+1}$.

Let (z, x, y) be the play along σ against $(G_0, G_1, \dots)$, then $(z, x, y) \in C$ and $y \notin \bigcup_{n\in\omega} G_n$. This contradicts to $\mathrm{proj}^2_0(\mathrm{proj}^3_0(C)) \subseteq \bigcup_{n\in\omega} G_n$.

So player I doesn't have a winning strategy. Therefore, by $\mathrm{Det}(\Gamma)$, player II has a winning strategy τ. Put

$$E = \bigcup\{G_n : (G_0, \dots, G_n) \text{ is the play along } \tau \text{ against some } (s_0, t_0, u_0 \dots, s_n, t_n, u_n)\}.$$

Then we have $\mathrm{proj}^2_0(\mathrm{proj}^3_0(B)) \subseteq E$. In order to check this, let $(z, x, y) \in B$. Consider the player I's play (z, x, y). Let $(G_0, G_1, \dots)$ be the play along τ against (z, x, y). Since II wins, $y \in \bigcup_{n\in\omega} G_n \subseteq E$.

Also we have

$$\mu(E) \leq \sum_n 8^{n+1} \frac{\varepsilon}{16^{n+1}} = \varepsilon.$$

Therefore we have $\mu^*(\mathrm{proj}^2_0(A)) \leq \mu(E) \leq \varepsilon$. $\square$

3.6 Consequences of large cardinals

In this section, using large cardinals, we separate $\mathsf{GP}(\boldsymbol{\Sigma}^1_{n+1})$ and $\mathsf{GP}(\boldsymbol{\Sigma}^1_n)$ for every $n \geq 2$.

For a pointclass Γ, recall that $\lhd$ is a Γ-good wellordering of the reals if it is a wellordering of the reals of order-type ω_1, it is in Γ and the relation $\{(x, y) : x \text{ codes the initial segment below } y \text{ with respect to } \lhd\}$ is in Γ.

Fact 3.6.1 ([BW97] and [Ste95]). (1) If ZFC is consistent, then so is ZFC plus $\boldsymbol{\Sigma}^1_2$ Lebesgue measurability plus "there is a $\boldsymbol{\Sigma}^1_3$ good wellordering of the reals of length ω_1".

(2) Assume that there are n many Woodin cardinals. Then there is an inner model M_n of ZFC that models $\mathrm{Det}(\boldsymbol{\Sigma}^1_n)$ and "there is a $\boldsymbol{\Sigma}^1_{n+2}$ good wellordering of the reals".

Lemma 3.6.2. Let $n \geq 2$. If there is a $\boldsymbol{\Sigma}^1_n$ good wellordering $\unlhd$ of the reals of length ω_1, then there is a cofinal increasing sequence of ω^ω whose image is $\boldsymbol{\Delta}^1_n$.

Proof. We define a function $G\colon [\omega^\omega]^{\leq\aleph_0} \to \omega^\omega$ by

$$G(S) = \unlhd\text{-minimum } x \text{ such that } x \text{ dominates all elements in } S$$

We define a sequence $(x_\alpha : \alpha < \omega_1)$ of reals in ω^ω by

$$x_\alpha = G(\{x_\beta : \beta < \alpha\}) \text{ (for } \alpha < \omega_1).$$

Then we put

$$D = \{x_\alpha : \alpha < \omega_1\}.$$

First we claim that D is $\boldsymbol{\Sigma}^1_n$. Using the usual technique that writes a recursive construction in the way of the existence of an approximation, we have

$$\begin{aligned} x \in D \iff &(\exists\alpha < \omega_1)(\exists F\colon \alpha + 1 \to \omega^\omega) \\ &[(\forall\beta < \alpha)(F(\beta) = G(F \restriction \beta)) \mathbin{\&} x = F(\alpha)]. \end{aligned}$$

Eliminating ordinal variables, we have

$$\begin{aligned} x \in D \iff & (\exists z)(\exists w)(\exists f\colon \omega \to \omega^\omega) \\ & [w \text{ codes the initial segment below } z \ \& \\ & (\forall k)(\exists w')[w' \text{ codes the initial segment below } w(k) \ \& \\ & f(k) = G(\{f(i) : \exists j \ ((w)_i = (w')_j)\})] \ \& \\ & x = G(\operatorname{ran} f)]. \end{aligned}$$

Note that the expression $x = G(\operatorname{ran} f)$ can be written as a $\mathbf{\Sigma}^1_n$ formula since

$$\begin{aligned} x = G(\operatorname{ran} f) \iff & (\forall k)[f(k) \leq^* x \ \& \\ & \exists w(w \text{ codes the initial segment below } x \ \& \ (\forall m)\neg(\forall k) f(k) \leq^* (w)_m)] \end{aligned}$$

Similarly, the expression $f(k) = G(\{f(i) : \exists j \ ((w)_i = (w')_j)\})$ can be also written as a $\mathbf{\Sigma}^1_n$ formula. Therefore D is $\mathbf{\Sigma}^1_n$.

Next, we show that D is $\mathbf{\Pi}^1_n$. As with the above claim, we have

$$\begin{aligned} x \notin D \iff & (\exists \alpha < \omega_1)(\exists F\colon \alpha + 1 \to \omega^\omega) \\ & [(\forall \beta < \alpha)(F(\beta) = G(F \restriction \beta)) \ \& \ x \leq^* F(\alpha) \ \& \ (\forall \beta < \alpha)(x \neq F(\beta))]. \end{aligned}$$

By the same transform of formulas in the above claim, we have that $\neg D$ is $\mathbf{\Sigma}^1_n$. $\square$

Lemma 3.6.3. Let $n \geq 2$. If there is a $\mathbf{\Sigma}^1_n$ good wellordering $\trianglelefteq$ of the reals of length ω_1, then $\neg\mathsf{GP}(\mathbf{\Delta}^1_n)$ holds.

Proof. Let D denote the set defined in Lemma 3.6.2. We define a set A by

$$A = \{(x, y) \in \omega^\omega \times \omega^\omega : y \trianglelefteq z \text{ for the minimum } z \in D \text{ that dominates } x\}.$$

Then we have

$$\begin{aligned} (x, y) \in A \iff & (\exists z)(\exists w) \\ & [w \text{ codes the initial segment below } z \ \& \ z \in D \ \& \\ & x \leq^* z \ \& \ (\forall k)((w)_k \in D \to x \not\leq^* (w)_k) \ \& \\ & y \trianglelefteq z] \end{aligned}$$

So A is $\mathbf{\Sigma}^1_n$. Moreover, since we have

$$\begin{aligned} (x, y) \in A \iff & (\forall z)(\forall w) \\ & [[w \text{ codes the initial segment below } z \ \& \ z \in D \ \& \\ & x \leq^* z \ \& \ (\forall k)((w)_k \in D \to x \not\leq^* (w)_k)] \ \to \\ & y \trianglelefteq z], \end{aligned}$$

it is also true that A is $\mathbf{\Pi}^1_n$. So A is $\mathbf{\Delta}^1_n$.

Since each A_x is countable and $\bigcup_{x \in \omega^\omega} A_x = \omega^\omega$, this A witnesses $\neg\mathsf{GP}(\mathbf{\Delta}^1_n)$. $\square$

Corollary 3.6.4. (1) If ZFC is consistent, then so is $\mathsf{ZFC} + \mathsf{GP}(\boldsymbol{\Sigma}^1_2) + \neg\mathsf{GP}(\boldsymbol{\Delta}^1_3)$.

(2) For every $n \geq 1$, if ZFC + (there are n many Woodin cardinals) is consistent, then so is $\mathsf{ZFC} + \mathsf{GP}(\boldsymbol{\Sigma}^1_{n+1}) + \neg\mathsf{GP}(\boldsymbol{\Delta}^1_{n+2})$.

Proof. As for (1), combine Fact 3.6.1 (1), Theorem 3.1.3 and Lemma 3.6.3. To show (2), combine Fact 3.6.1 (2), Theorem 3.5.1 and Lemma 3.6.3. □

3.7 $\mathsf{GP}(\mathsf{all})$ in Solovay models

Now that we know that AD implies $\mathsf{GP}(\mathsf{all})$, it is natural to ask whether $\mathsf{GP}(\mathsf{all})$ holds in Solovay models. In this section, we will solve this question affirmatively.

Basic information about Solovay models can be found in [Kan08, Chapter 3].

Let us recall that Coll_κ denotes the Levy collapse.

Definition 3.7.1. (1) $L(\mathbb{R})^M$ is a Solovay model over V (in the usual sense) if $M = V[G]$ for some inaccessible cardinal κ and $(V, \mathrm{Coll}_\kappa)$ generic filter G.

(2) $L(\mathbb{R})^M$ is a Solovay model over V in the weak sense if the following 2 conditions hold in M:

(a) For every $x \in \mathbb{R}$, ω_1 is an inaccessible cardinal in $V[x]$.

(b) For every $x \in \mathbb{R}$, $V[x]$ is a generic extension of V by some poset in V, which is countable in M.

Fact 3.7.2 (Woodin, see [BB04, Lemma 1.2])**.** If $L(\mathbb{R})^M$ is a Solovay model over V in the weak sense then there is a forcing poset $\mathbb{W}$ in M such that $\mathbb{W}$ adds no new reals and

$$\mathbb{W} \Vdash \text{“}L(\mathbb{R})^M \text{ is a Solovay model over } V \text{ (in the usual sense)”}.$$

Fact 3.7.3 ([BB04, Theorem 2.4])**.** Suppose that $L(\mathbb{R})^M$ is a Solovay model over V in the weak sense and $\mathbb{P}$ is a strongly-$\boldsymbol{\Sigma}^1_3$ absolutely-ccc poset in M. Let G be a $(M, \mathbb{P})$ generic filter. Then $L(\mathbb{R})^{M[G]}$ is also a Solovay model in V in the weak sense.

We don't define the terminology “strongly-$\boldsymbol{\Sigma}^1_3$ absolutely-ccc poset” here. But the random forcing is such a poset and we will use only the random forcing when applying Fact 3.7.3.

Lemma 3.7.4. Let M, N be models satisfying $V \subseteq M \subseteq N$. Assume that the $L(\mathbb{R})$ of each of M and N are Solovay models over V in the weak sense. Then for every formula $\varphi(v)$ in the language of set theory $\mathcal{L}_\in = \{\in\}$ and real r in M, the assertion “$L(\mathbb{R}) \models \varphi(r)$” is absolute between M and N.

Proof. By Fact 3.7.2, we may assume that $L(\mathbb{R})^M$ and $L(\mathbb{R})^N$ are Solovay models over V in the usual sense. By universality and homogeneity of the Levy collapse, we have

$$\begin{aligned} M \models \text{“}L(\mathbb{R}) \models \varphi(r)\text{”} &\iff V[r] \models [\mathrm{Coll}_\kappa \Vdash \text{“}L(\mathbb{R}) \models \varphi(r)\text{”}] \\ &\iff N \models \text{“}L(\mathbb{R}) \models \varphi(r)\text{”} \end{aligned}$$

□

Theorem 3.7.5. Let κ be an inaccessible cardinal and G be a $(V, \mathrm{Coll}_\kappa)$ generic filter. Then $L(\mathbb{R})^{V[G]}$ satisfies $\mathsf{GP}(\mathsf{all})$. That is, every Solovay model satisfies $\mathsf{GP}(\mathsf{all})$.

Proof. Let $A \subseteq \omega^\omega \times 2^\omega$ in $L(\mathbb{R})^{V[G]}$. Take a formula φ and an ordinal α such that

$$A = \{(x, y) : \varphi(\alpha, x, y)\}^{L(\mathbb{R})^{V[G]}}.$$

In $L(\mathbb{R})^{V[G]}$, assume that

(1) $(\forall x \in \omega^\omega)(\exists c_x \in \omega^\omega)(c_x$ is a Borel code for a measure 0 set & $A_x \subseteq \hat{c_x})$

(2) $(\forall x, x' \in \omega^\omega)(x \leq x' \to A_x \subseteq A_{x'})$.

Using the axiom of choice in $V[G]$ we can choose such a family $(c_x : x \in \omega^\omega)$. Note that this family is not necessarily in $L(\mathbb{R})^{V[G]}$.

Since every set of reals is measurable in $L(\mathbb{R})^{V[G]}$, $\bigcup_{x\in\omega^\omega} A_x$ is measurable in $L(\mathbb{R})^{V[G]}$. Now we assume that the measure is positive and take a closed code d in $L(\mathbb{R})^{V[G]}$ such that $\mu(\hat{d}) > 0$ and $\hat{d} \subseteq \bigcup_{x\in\omega^\omega} A_x$ in $L(\mathbb{R})^{V[G]}$.

Take a random real r over $V[G]$ with $r \in \hat{d}$. Then by Lemma 3.7.4, we have in $L(\mathbb{R})^{V[G][r]}$

(1') $(\forall x \in \omega^\omega \cap L(\mathbb{R})^{V[G]})(A_x \subseteq \hat{c_x})$, and

(2') $(\forall x, x' \in \omega^\omega)(x \leq x' \to A_x \subseteq A_{x'})$.

By randomness, we have $r \notin \hat{c_x}$ for all $x \in \omega^\omega \cap L(\mathbb{R})^{V[G]}$. But (2') and the fact that the random forcing is ω^ω-bounding imply $r \notin A_x$ for all $x \in \omega^\omega$ in $V[G][r]$. Thus we have $\hat{d} \smallsetminus \bigcup_{x\in\omega^\omega} A_x \neq \varnothing$ in $L(\mathbb{R})^{V[G][r]}$. Then using Lemma 3.7.4 again, we have $\hat{d} \smallsetminus \bigcup_{x\in\omega^\omega} A_x \neq \varnothing$ in $L(\mathbb{R})^{V[G]}$. It is a contradiction to the choice of d. □

3.8 A necessary condition for $\mathsf{GP}(\mathbf{\Delta}^1_2)$

From now on, we again assume ZFC.

As mentioned in Section 3.1, a sufficient condition for $\mathsf{GP}(\mathbf{\Sigma}^1_2)$ is every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable, or equivalently for every real a, there is an amoeba real over $L[a]$. (This equivalence was proved by Solovay, see [BJ95, Theorem 9.3.1]). In this section we give a necessary condition for $\mathsf{GP}(\mathbf{\Delta}^1_2)$.

Fact 3.8.1 (Spector–Gandy, see [CY15, Propositon 4.4.3])**.** Let A be a set of reals. Then A is a Σ^1_2 set iff there is a Σ_1 formula φ such that

$$x \in A \iff (L_{\omega_1^{L[x]}}[x], \in) \models \varphi(x).$$

The following is well-known.

Lemma 3.8.2. Let M be a model of ZFC contained by V. And assume that the set $\{y \in 2^\omega : y$ is a random real over $M\}$ has measure 1. Then there is a dominating real over M.

Proof. Let $\mathbf{nBC}$ denote the set of all Borel codes for measure 0 Borel sets. There is an absolute Tukey morphism (φ, ψ) that witnesses $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$. That is, (φ, ψ) satisfies $\varphi\colon \omega^\omega \to \mathbf{nBC}$, $\psi\colon \mathbf{nBC} \to \omega^\omega$, and $(\forall x \in \omega^\omega)(\forall y \in \mathbf{nBC})(\hat{\varphi(x)} \subseteq \hat{y} \to x \leq^* \psi(y))$. By absoluteness, if $x \in \omega^\omega \cap M$, then we have $\varphi(x) \in M$. Now

$$\bigcup_{x\in\omega^\omega\cap M} \hat{\varphi(x)}$$

has measure 0 since this is contained in $\{y \in 2^\omega : y \text{ is not a random real over } M\}$ by the definition of randomness. Take a $z \in \mathbf{nBC}$ such that

$$\bigcup_{x \in \omega^\omega \cap M} \hat{\varphi(x)} \subseteq \hat{z}.$$

Now put $w = \psi(z)$. Then using the fact that (φ, ψ) is Tukey morphism, we have w is a dominating real over M. □

Theorem 3.8.3. For every real a, $\mathsf{GP}(\Delta^1_2(a))$ implies there is a dominating real over $L[a]$. Thus, $\mathsf{GP}(\boldsymbol{\Delta}^1_2)$ implies for every real a, there is a dominating real over $L[a]$. In particular $V = L$ implies $\neg\mathsf{GP}(\Delta^1_2)$.

Proof. Fix a real a. Assume that $L[a] \cap \omega^\omega$ is unbounded. Note that, in this situation, we have $\omega_1^{L[a]} = \omega_1$. Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a cofinal increasing sequence in $\omega^\omega \cap L[a]$. We can take this sequence with a $\Delta_1(a)$ definition by using a $\Delta_1(a)$ canonical wellordering of $L[a] \cap \omega^\omega$. Note that this sequence is unbounded in $V \cap \omega^\omega$ by assumption.

Take a sequence $\langle c_\alpha : \alpha < \omega_1 \rangle$ consisting of all Borel codes for measure 0 Borel sets in $L[a]$. As above, we can take this sequence with a $\Delta_1(a)$ definition.

For each $x \in \omega^\omega$, put

$$\alpha(x) = \min\{\alpha : x_\alpha \not\leq^* x\}.$$

This is well-defined since $\langle x_\alpha : \alpha < \omega_1 \rangle$ is unbounded in $V \cap \omega^\omega$. Also put

$$A_x = \bigcup_{\beta < \alpha(x)} \hat{c_\beta}.$$

Then the set A is $\Delta^1_2(a)$, by Spector–Gandy theorem and the following equations:

$$\begin{aligned} A &= \{(x, y) \in \omega^\omega \times 2^\omega : (\exists \beta < \alpha(x))\ y \in \hat{c_\beta}\} \\ &= \{(x, y) \in \omega^\omega \times 2^\omega : (\exists \alpha)(x_\alpha \not\leq^* x \ \&\ (\forall \beta < \alpha)(x_\beta \leq^* x) \ \&\ (\exists \beta < \alpha)(y \in \hat{c_\beta}))\} \\ &= \{(x, y) \in \omega^\omega \times 2^\omega : (\forall \alpha)((x_\alpha \not\leq^* x \ \&\ (\forall \beta < \alpha)(x_\beta \leq^* x)) \to (\exists \beta < \alpha)(y \in \hat{c_\beta}))\}. \end{aligned}$$

Note that each A_x ($x \in \omega^\omega$) is a measure 0 set since it is a countable union of measure 0 sets. And we can easily observe that $x \leq^* x'$ implies $A_x \subseteq A_{x'}$.

Since $\alpha \leq \alpha(x_\alpha)$, we have $\bigcup_{x \in \omega^\omega} A_x = \bigcup_{\alpha < \omega_1} \hat{c_\alpha}$. Thus it is sufficient to show that $C := \bigcup_{\alpha < \omega_1} \hat{c_\alpha}$ is not a measure 0 set. In order to show this, assume that C is a measure 0 set. Note that if a real $y \in 2^\omega$ does not belong to C, then y is a random real over $L[a]$ since the sequence $\langle c_\alpha : \alpha < \omega_1 \rangle$ enumerates all measure 0 Borel codes in $L[a]$. So since we assumed C is measure 0, the set $\{y \in 2^\omega : y \text{ is a random real over } L[a]\}$ has measure 1. Thus by Lemma 3.8.2, there is a dominating real over $L[a]$. This contradicts the assumption.

So we constructed a set A that violates $\mathsf{GP}(\boldsymbol{\Delta}^1_2)$. This finishes the proof. □

Therefore, we obtain the following diagram of implications.

$$
\begin{tikzcd}
\boldsymbol{\Sigma}^1_2\text{-LM} \arrow[rr] \arrow[dr] & & \boldsymbol{\Sigma}^1_2\text{-BP} \arrow[rr] & & \begin{array}{l}(\forall a \in \mathbb{R})(\exists z \in \omega^\omega) \\ \quad (z \text{ is a dominating real over } L[a])\end{array} \\
& \mathsf{GP}(\boldsymbol{\Sigma}^1_2) \arrow[rr] & & \mathsf{GP}(\boldsymbol{\Delta}^1_2) \arrow[ur] &
\end{tikzcd}
$$

Here, $\boldsymbol{\Sigma}^1_2$-LM means $\boldsymbol{\Sigma}^1_2$-Lebesgue measurability and $\boldsymbol{\Sigma}^1_2$-BP means $\boldsymbol{\Sigma}^1_2$-Baire property. $\boldsymbol{\Sigma}^1_2$-LM and $\mathsf{GP}(\boldsymbol{\Sigma}^1_2)$ can be separated since the Laver model over L satisfies $\mathsf{GP}(\mathsf{all})$ but not $\boldsymbol{\Sigma}^1_2$-LM.

3.9 Open problems

The following open problems remain.

Problem 3.9.1. (1) Is $\mathsf{ZFC} + (\mathfrak{c} > \aleph_2) + \mathsf{GP}(\mathsf{all})$ consistent?

(2) Is $\mathsf{ZFC} + (\mathfrak{b} < \mathfrak{d}) + \mathsf{GP}(\mathsf{all})$ consistent?

(3) Can $\mathsf{GP}(\boldsymbol{\Sigma}^1_2)$ and $\mathsf{GP}(\boldsymbol{\Delta}^1_2)$ be separated?

(4) Can $\mathsf{GP}(\boldsymbol{\Delta}^1_2)$ and $(\forall a \in \mathbb{R})(\exists z \in \omega^\omega)(z$ is a dominating real over $L[a])$ be separated?

(5) Is there a model of ZF satisfying that every set of reals is measurable and $\neg\mathsf{GP}(\mathsf{all})$?

(6) Is it possible to separate $\mathsf{GP}(\boldsymbol{\Sigma}^1_{n+1})$ and $\mathsf{GP}(\boldsymbol{\Sigma}^1_n)$ for some (or every) $n \geq 3$ without large cardinals?

Chapter 4

Hausdorff measures

Hausdorff measures are important tools for measuring Lebesgue null sets. We study cardinal invariants determined by the Hausdorff measure zero ideals $\mathcal{N}^f$ for gauge functions f. In particular, we study cardinal invariants of the s-dimensional Hausdorff measure zero ideal $\mathcal{N}^s$ for $s > 0$ and cardinal invariants of the Hausdorff dimension zero ideal HDZ.

A classical result by Besicovitch in [Bes33] is that the strong measure zero ideal $\mathcal{SN}$ is the intersection of all Hausdorff measure zero ideals $\mathcal{N}^f$. On the other hand, Yorioka introduced Yorioka ideals in [Yor02] to analyze the strong measure zero ideal $\mathcal{SN}$ and showed that their intersection is also equal to $\mathcal{SN}$. Our main results are the relationship between Hausdorff measure zero ideals and Yorioka ideals (Section 4.2) and using this and prior studies ([KM22; OK14]) on Yorioka ideals, the separation of many covering numbers and many uniformity number of Hausdorff measure zero ideals (Section 4.5 and 4.6).

As for other sections, in Section 4.1, we show cardinal invariants of the ideal HDZ do not change if we change the underlying metric space from the Cantor space to the Euclidean space $\mathbb{R}^d$. In Section 4.3, we consider the additivity and the cofinality of HDZ. In Section 4.4, we separate the uniformity of the null ideal and the uniformity of the s-dimensional Hausdorff measure 0 ideal $\mathcal{N}^s$ using the Mathias forcing, which simplifies the proof in [SS05]. In Section 4.7, we consider Goldstern's principle of Hausdorff measures, which generalizes the discussion in Chapter 3. In Section 4.8, we show that Laver forcing preserves Hausdorff measures. Lastly, in Section 4.9, we introduce an amoeba type forcing of Hausdorff measures.

Only in this chapter we use the following notation.

Definition 4.0.1. For functions $f, g\colon \omega \to \mathbb{R}$, define $f+g, f-g, f\cdot g, f/g, f^g$ and $\sqrt{f}$ as follows:

$$\begin{aligned}
(f+g)(n) &= f(n)+g(n),\\
(f-g)(n) &= f(n)-g(n),\\
(f\cdot g)(n) &= f(n)\cdot g(n),\\
(f/g)(n) &= f(n)/g(n),\\
(f^g)(n) &= f(n)^{g(n)},\\
(\sqrt{f})(n) &= \sqrt{f(n)}.
\end{aligned}$$

id denotes the identity function from a set into itself. We often use this notation when the domain is

ω. We use the notation 2 as the constant function returning 2.

For functions $f, g\colon \omega \to \omega$,let $f^{\triangledown g}(n) = |[f(n)]^{\leq g(n)}|$.

In the rest of this section, we review basic definitions of Hausdorff measures.

Definition 4.0.2. A function $f\colon [0,\infty) \to [0,\infty)$ is a *gauge function* if $f(0) = 0, \lim_{x\to 0} f(x) = 0$ and f is nondecreasing.

Let X be a metric space. For $A \subseteq X$, f a gauge function and $\delta \in (0,\infty]$, we define

$$\mathcal{H}^f_\delta(A) = \inf\{\sum_{n=0}^{\infty} f(\mathrm{diam}(C_n)) : C_n \subseteq X \text{ (for } n \in \omega) \text{ with } A \subseteq \bigcup_{n\in\omega} C_n \text{ and } (\forall n)(\mathrm{diam}(C_n) \leq \delta)\}.$$

Next, for $A \subseteq X$ and f a gauge function, we define

$$\mathcal{H}^f(A) = \lim_{\delta\to 0} \mathcal{H}^f_\delta(A).$$

We call $\mathcal{H}^f(A)$ the *Hausdorff measure* with gauge function f. In particular, for $A \subseteq X$ and $s > 0$, let

$$\mathcal{H}^s(A) = \mathcal{H}^{\mathrm{pow}_s}(A)$$

where $\mathrm{pow}_s(x) = x^s$. For $A \subseteq X$, let

$$\dim_{\mathrm{H}}(A) = \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

We call $\dim_{\mathrm{H}}(A)$ the *Hausdorff dimension* of A.

We metrize the Cantor space 2^ω by

$$d(x,y) = \begin{cases} 0 & \text{(if } x = y), \\ 2^{-\min\{n : x(n)\neq y(n)\}} & \text{(otherwise).} \end{cases}$$

Definition 4.0.3. (1) For a metric space X, define $\mathsf{HDZ}_X = \{A \subseteq X : \dim_{\mathrm{H}}(A) = 0\}$.

(2) Define $\mathsf{HDZ} = \mathsf{HDZ}_{2^\omega}$.

Definition 4.0.4. For a metric space X and a gauge function f, define $\mathcal{N}^f_X = \{A \subseteq X : \mathcal{H}^f(A) = 0\}$. Especially we define $\mathcal{N}^f = \mathcal{N}^f_{2^\omega}$. For $s > 0$, define $\mathcal{N}^s_X = \mathcal{N}^{\mathrm{pow}_s}_X$ and $\mathcal{N}^s = \mathcal{N}^{\mathrm{pow}_s}_{2^\omega}$.

Remark 4.0.5. (1) $\mathcal{N}^1 = \mathcal{N}$.

(2) $\mathsf{HDZ} = \bigcap_{s>0} \mathcal{N}^s$.

Definition 4.0.6. For $\sigma \in (2^{<\omega})^\omega$, define $\mathrm{ht}\sigma\colon \omega \to \omega$ and $[\sigma]_\infty \subseteq 2^\omega$ by

$$(\mathrm{ht}\sigma)(n) = |\sigma(n)| \text{ and}$$

$$[\sigma]_\infty = \{x \in 2^\omega : (\exists^\infty n)\sigma(n) \subseteq x\}.$$

For $g \in \omega^\omega$, define

$$\mathcal{J}_g = \{A \subseteq 2^\omega : (\exists\sigma \in (2^{<\omega})^\omega)(\mathrm{ht}\sigma = g \;\&\; A \subseteq [\sigma]_\infty)\}.$$

For $f, g \in \omega^\omega$, define

$$f \ll g \iff (\forall k \in \omega)(f \circ \mathrm{pow}_k \leq^* g)$$

For $f \in \omega^\omega$ increasing, define

$$\mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g.$$

We call $\mathcal{I}_f$ the Yorioka ideal for f.

4.1 Stability under changing underlying spaces

In this section, we prove that cardinal invariants of the ideal HDZ do not change if we change the underlying metric space from the Cantor space to the Euclidean space $\mathbb{R}^d$.

Definition 4.1.1. Let X and Y be metric spaces and $\alpha, c > 0$. A map $f\colon X \to Y$ is said to be *α-Hölder* with constant c if for all $x, x' \in X$ we have $d(f(x), f(x')) \leq c \cdot d(x, x')^\alpha$. A map $f\colon X \to Y$ is said to be *α-co-Hölder* with constant c if for all $x, x' \in X$ we have $d(f(x), f(x')) \geq c \cdot d(x, x')^\alpha$. A map $f\colon X \to Y$ is said to be *α-bi-Hölder* with constants c_1, c_2 if it is both α-Hölder with constant c_1 and α-co-Hölder with constant c_2.

Proposition 4.1.2. Let X and Y be metric spaces and $\alpha > 0$.

(1) If there is α-Hölder map $f\colon X \to Y$ with constant c, then for all $s > 0$ we have $\mathcal{H}^{s/\alpha}(f(X)) \leq c^{s/\alpha}\mathcal{H}^s(X)$ and $\dim_{\mathrm{H}} f(X) \leq (1/\alpha)\dim_{\mathrm{H}} X$.

(2) If there is α-co-Hölder map $f\colon X \to Y$ with constant c, then for all $s > 0$ we have $\mathcal{H}^{s/\alpha}(f(X)) \geq c^{s/\alpha}\mathcal{H}^s(X)$ and $\dim_{\mathrm{H}} f(X) \geq (1/\alpha)\dim_{\mathrm{H}} X$.

(3) If there is α-bi-Hölder map $f\colon X \to Y$ with constant c_1, c_2, then for all $s > 0$ we have $c_1^{s/\alpha}\mathcal{H}^s(X) \leq \mathcal{H}^{s/\alpha}(f(X)) \leq c_2^{s/\alpha}\mathcal{H}^s(X)$ and $\dim_{\mathrm{H}} f(X) = (1/\alpha)\dim_{\mathrm{H}} X$.

Proof. Item 1. Let $\delta > 0$ and $\langle C_n : n \in \omega\rangle$ be a δ-cover of X, that is $X \subseteq \bigcup_n C_n$ and $\mathrm{diam}(C_n) \leq \delta$ for all n. Then $\langle f(C_n) : n \in \omega\rangle$ is a cover of $f(X)$ and the diameter of each member satisfies

$$\mathrm{diam}(f(C_n)) \leq c \cdot \mathrm{diam}(C_n)^\alpha \leq c \cdot \delta^\alpha =: \varepsilon.$$

So $\langle f(C_n) : n \in \omega\rangle$ is a ε-cover of $f(X)$. Thus

$$\mathcal{H}^{s/\alpha}_\varepsilon(f(X)) \leq \sum_n \mathrm{diam}(f(C_n))^{s/\alpha} \leq \sum_n c^{s/\alpha} \cdot \mathrm{diam}(C_n)^s.$$

Take the infimum for (C_n) we get the following.

$$\mathcal{H}^{s/\alpha}_\varepsilon(f(X)) \leq c^{s/\alpha}\mathcal{H}^s_\delta(X).$$

Letting δ tend to 0, we have

$$\mathcal{H}^{s/\alpha}(f(X)) \leq c^{s/\alpha}\mathcal{H}^s(X).$$

In order to prove the dimension inequality, Let $s > \dim_{\mathrm{H}} X$. Then $\mathcal{H}^s(X) = 0$, so $\mathcal{H}^{s/\alpha}(f(X))$ is also equal to 0. Thus $s/\alpha \geq \dim_{\mathrm{H}} f(X)$.

Item 2. Observe that every α-co-Hölder map $f\colon X \to Y$ with constant c is injective and the inverse map $f^{-1}\colon f(X) \to X$ is $(1/\alpha)$-Hölder map with constant $c^{-1/\alpha}$ and use item 1.

Item 3. Combine Item 1 and 2. □

Proposition 4.1.3. Let X, Y, X', Y' be metric spaces and $\alpha > 0$.

(1) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are α-Hölder maps, then $f \times g\colon X \times Y \to X' \times Y'$ is also an α-Hölder map.

(2) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are α-co-Hölder maps, then $f \times g\colon X \times Y \to X' \times Y'$ is also an α-co-Hölder map.

Proof. We now adopt the max metric as a metric of product space:

$$d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\} \ (x_1, x_2 \in X, y_1, y_2 \in Y).$$

Note that the above metric and other two metrics $d(x_1, x_2) + d(y_1, y_2)$ and $\sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}$ are Lipschitz equivalent.

By the assumption, there are $c_1, c_2 > 0$ such that

$$\begin{aligned} d(f(x_1), f(x_2)) &\leq c_1 d(x_1, x_2)^\alpha, \\ d(g(y_1), g(y_2)) &\leq c_2 d(y_1, y_2)^\alpha. \end{aligned}$$

Then, we have

$$\max\{d(f(x_1), f(x_2)), d(g(y_1), g(y_2))\} \leq \max\{c_1, c_2\} \max\{d(x_1, x_2), d(y_1, y_2)\}^\alpha.$$

So item (1) is proven. Item (2) can be shown by the same argument. $\square$

Proposition 4.1.4. For every $\alpha \in (1, \infty)$, there is a α-co-Hölder map $f\colon 2^\omega \to [0, 1]$.

Proof. Put $\beta = 2^{-\alpha}$. Then we have $0 < \beta < 1/2$. Define $f\colon 2^\omega \to [0, 1]$ by

$$f(x) = (1 - \beta) \sum_{n \in \omega} \beta^n x(n).$$

Let $x \neq y \in 2^\omega$ and $n_0 = \min\{n \in \omega : x(n) \neq y(n)\}$. Then

$$\begin{aligned} d(f(x), f(y)) &= (1 - \beta) \left| \sum_{n \in \omega} \beta^n (x(n) - y(n)) \right| \\ &\geq (1 - \beta) \left(|\beta^{n_0}| - \left| \sum_{n > n_0} \beta^n (x(n) - y(n)) \right| \right) \\ &\geq (1 - \beta)(\beta^{n_0} - \beta^{n_0 + 1}/(1 - \beta)) \\ &= (1 - 2\beta)\beta^{n_0} \\ &= (1 - 2\beta)(2^{-n_0})^{-\log_2 \beta} \\ &= (1 - 2\beta)(d(x, y))^\alpha. \end{aligned}$$

$\square$

Proposition 4.1.5. There is a 1-co-Hölder map $f\colon [0, 1] \to 2^\omega$.

Proof. Define $g\colon 2^\omega \to [0, 1]$ by

$$g(x) = \sum_{n \in \omega} \frac{x(n)}{2^{n+1}}.$$

Let $f\colon [0,1] \to 2^\omega$ satisfies $g \circ f = \mathrm{id}$. In order to show f is a 1-co-Hölder map, it suffices to prove that

$$(\forall x, y \in 2^\omega)(d(g(x), g(y)) \leq d(x, y)).$$

Fix $x, y \in 2^\omega$. If $x = y$, then it is obvious. So assume that $x \neq y$ and let n_0 be the minimum n that $x(n) \neq y(n)$. Then

$$\begin{aligned}
d(g(x), g(y)) &= \left| \sum_{n \in \omega} \frac{x(n) - y(n)}{2^{n+1}} \right| \\
&\leq \frac{1}{2^{n_0+1}} + \sum_{n > n_0} \frac{1}{2^{n+1}} \\
&= 1/2^{n_0} \\
&= d(x, y).
\end{aligned}$$

□

Fact 4.1.6. For every $d \in \omega \setminus \{0\}$, we have $\dim_{\mathrm{H}}([0,1]^d) = d$.

Proposition 4.1.7. $\dim_{\mathrm{H}}(2^\omega) = 1$.

Proof. By Proposition 4.1.5,

$$\dim_{\mathrm{H}}(2^\omega) \geq \dim_{\mathrm{H}}[0,1] = 1.$$

On the other hand, for every $\alpha > 1$, by Proposition 4.1.4,

$$\dim_{\mathrm{H}}(2^\omega) \leq \alpha \dim_{\mathrm{H}}[0,1] = \alpha.$$

So $\dim_{\mathrm{H}}(2^\omega) \leq 1$.

□

Proposition 4.1.8. For every $d \in \omega \setminus \{0\}$, there is a $(1/d)$-bi-Hölder map $f\colon 2^\omega \to (2^\omega)^d$.

Proof. Define f by

$$f(x)(i)(m) = x(m \cdot d + i) \text{ (for } x \in 2^\omega, i < d \text{ and } m \in \omega).$$

Let $x \neq y \in 2^\omega$ and $n_0 = \min\{n \in \omega : x(n) \neq y(n)\}$. And take $i_0 < n_0$ and $m_0 \in \omega$ such that $n_0 = m_0 \cdot d + i_0$. Then $f(x)(i_0)(m_0) \neq f(y)(i_0)(m_0)$ and $f(x)(i)(m) = f(y)(i)(m)$ for any $i < n_0$ and $m < m_0$. So

$$d(f(x), f(y)) = 2^{-m_0}.$$

Now we have

$$dm_0 \leq n_0 \leq d(m_0 + 1)$$

So

$$n_0/d - 1 \leq m_0 \leq n_0/d.$$

Thus

$$d(x, y)^{1/d} \leq d(f(x), f(y)) \leq 2d(x, y)^{1/d}.$$

□

Proposition 4.1.9. Let $f\colon X \to Y$ be an α-co-Hölder map for some $\alpha > 0$. Then

$$\mathrm{non}(\mathsf{HDZ}_X) \geq \mathrm{non}(\mathsf{HDZ}_Y) \text{ and } \mathrm{cov}(\mathsf{HDZ}_X) \leq \mathrm{cov}(\mathsf{HDZ}_Y).$$

Proof. Define a Tukey morphism $(f, f^{-1}) : \mathbf{Cov}(\mathsf{HDZ}_X) \to \mathbf{Cov}(\mathsf{HDZ}_Y)$. Since f^{-1} is $1/\alpha$-Hölder, this satisfies

$$A \in \mathsf{HDZ}_Y \Rightarrow f^{-1}(A) \in \mathsf{HDZ}_X.$$ □

Lemma 4.1.10. For every $d \in \omega \smallsetminus \{0\}$, $\mathrm{non}(\mathsf{HDZ}_{[0,1]^d}) = \mathrm{non}(\mathsf{HDZ}_{\mathbb{R}^d})$ and $\mathrm{cov}(\mathsf{HDZ}_{[0,1]^d}) = \mathrm{cov}(\mathsf{HDZ}_{\mathbb{R}^d})$.

Proof. Since $[0,1]^d \subseteq \mathbb{R}^d$, it is clear that $\mathrm{non}(\mathsf{HDZ}_{\mathbb{R}^d}) \leq \mathrm{non}(\mathsf{HDZ}_{[0,1]^d})$ and $\mathrm{cov}(\mathsf{HDZ}_{[0,1]^d}) \leq \mathrm{cov}(\mathsf{HDZ}_{\mathbb{R}^d})$.

Now we show $\mathrm{non}(\mathsf{HDZ}_{[0,1]^d}) \leq \mathrm{non}(\mathsf{HDZ}_{\mathbb{R}^d})$. Let $A \notin \mathsf{HDZ}_{\mathbb{R}^d}$. Then by σ-additivity of Hausdorff measures, we have $A \cap [-n,n]^d \notin \mathsf{HDZ}_{[-n,n]^d}$ for some $n \in \omega$. Then $|A \cap [-n,n]^d| \geq \mathrm{non}(\mathsf{HDZ}_{[-n,n]^d}) = \mathrm{non}(\mathsf{HDZ}_{[-1,1]^d})$. So $|A| \geq \mathrm{non}(\mathsf{HDZ}_{[-1,1]^d})$. Thus $\mathrm{non}(\mathsf{HDZ}_{\mathbb{R}^d}) \geq \mathrm{non}(\mathsf{HDZ}_{[0,1]^d})$.

Next we show $\mathrm{cov}(\mathsf{HDZ}_{\mathbb{R}^d}) \leq \mathrm{cov}(\mathsf{HDZ}_{[0,1]^d})$. Take $\mathcal{F} \subseteq \mathsf{HDZ}_{[0,1]^d}$ of size $\mathrm{cov}(\mathsf{HDZ}_{[0,1]^d})$ such that $\bigcup \mathcal{F} = [0,1]^d$. Define

$$\mathcal{G} = \{ \bigcup_{n \in \omega} \mathrm{scale}_n \text{“}(X) : X \in \mathcal{F} \},$$

where $\mathrm{scale}_n \colon [0,1]^d \to [-n,n]^d; \boldsymbol{x} \mapsto 2n\boldsymbol{x} - (n, n, \dots, n)$. Then $\mathcal{G}$ is a subset of $\mathsf{HDZ}_{\mathbb{R}^d}$ and of size $\leq \mathrm{cov}(\mathsf{HDZ}_{[0,1]^d})$ and satisfies $\bigcup \mathcal{G} = \mathbb{R}^d$. □

Theorem 4.1.11. For all $d \in \omega \smallsetminus \{0\}$, $\mathrm{non}(\mathsf{HDZ}) = \mathrm{non}(\mathsf{HDZ}_{\mathbb{R}^d})$ and $\mathrm{cov}(\mathsf{HDZ}) = \mathrm{cov}(\mathsf{HDZ}_{\mathbb{R}^d})$.

Proof. By Lemma 4.1.10, it suffices to show that $\mathrm{non}(\mathsf{HDZ}) = \mathrm{non}(\mathsf{HDZ}_{[0,1]^d})$ and $\mathrm{cov}(\mathsf{HDZ}) = \mathrm{cov}(\mathsf{HDZ}_{[0,1]^d})$.

By Proposition 4.1.5, there is a 1-co-Hölder $[0,1] \to 2^\omega$. Then by Proposition 4.1.3, there is a 1-co-Hölder $[0,1]^d \to (2^\omega)^d$. By Proposition 4.1.8, there is a d-bi-Hölder $(2^\omega)^d \to 2^\omega$. Composing these maps, we obtain d-co-Hölder map $[0,1]^d \to 2^\omega$. So by Proposition 4.1.9, we have $\mathrm{non}(\mathsf{HDZ}_{[0,1]^d}) \geq \mathrm{non}(\mathsf{HDZ})$ and $\mathrm{cov}(\mathsf{HDZ}_{[0,1]^d}) \leq \mathrm{cov}(\mathsf{HDZ})$.

On the other hand by Proposition 4.1.4, there is a 2-co-Hölder map $2^\omega \to [0,1]$. Then by Proposition 4.1.3, there is a 2-co-Hölder $(2^\omega)^d \to ([0,1])^d$. By Proposition 4.1.8, there is a $(1/d)$-bi-Hölder $2^\omega \to (2^\omega)^d$. Composing these maps, we obtain $(2/d)$-co-Hölder map $2^\omega \to [0,1]^d$. So by Proposition 4.1.9, we have $\mathrm{non}(\mathsf{HDZ}) \geq \mathrm{non}(\mathsf{HDZ}_{[0,1]^d})$ and $\mathrm{cov}(\mathsf{HDZ}) \leq \mathrm{cov}(\mathsf{HDZ}_{[0,1]^d})$. □

Conjecture 4.1.12. (1) For every compact Polish space X with $0 < \mathcal{H}^s(X) < \infty$ for some $s > 0$, $\mathrm{non}(\mathsf{HDZ}_X) = \mathrm{non}(\mathsf{HDZ})$ and $\mathrm{cov}(\mathsf{HDZ}_X) = \mathrm{cov}(\mathsf{HDZ})$.

4.2 Hausdorff measure zero ideals and Yorioka ideals

Yorioka ideals were introduced in order to analyze the strong measure zero ideal by Yorioka [Yor02]. Indeed, the intersection of all Yorioka ideals is equal to the strong measure zero ideal. On the other hand, as a classic result, Besicovitch showed in [Bes33] that the intersection of all Hausdorff measure zero ideals is also equal to the strong measure zero ideal. In this section, we investigate the relation between Yorioka ideals and Hausdorff measure zero ideals.

Lemma 4.2.1. For a gauge function f and $A \subseteq X$, if $\mathcal{H}^f_\infty(A) = 0$, then $\mathcal{H}^f(A) = 0$.

Proof. By $\mathcal{H}^f_\infty(A) = 0$, we have

$$(\forall \varepsilon > 0)(\exists \langle C_n : n \in \omega\rangle)(A \subseteq \bigcup_n C_n \;\&\; \sum_n f(\mathrm{diam}(C_n)) \leq \varepsilon). \tag{4.1}$$

Let $\varepsilon, \delta > 0$. Take $\delta' < \delta$ such that $f(\delta') < f(\delta)$. Put $\varepsilon' = \min\{\varepsilon, f(\delta')\}$. Then by (4.1), we can take $\langle C_n : n \in \omega\rangle$ such that

$$\sum_n f(\mathrm{diam}(C_n)) \leq \varepsilon'.$$

Then for each n, we have

$$f(\mathrm{diam}(C_n)) \leq \varepsilon' \leq f(\delta') < f(\delta).$$

Since f nondecreasing we have

$$\mathrm{diam}(C_n) \leq \delta.$$

So we have showed

$$(\forall \varepsilon, \delta > 0)(\exists \langle C_n : n \in \omega\rangle)(A \subseteq \bigcup_n C_n \;\&\; \sum_n f(\mathrm{diam}(C_n)) \leq \varepsilon \;\&\; (\forall n)(\mathrm{diam}(C_n) \leq \delta)).$$

That is, we showed $\mathcal{H}^f(A) = 0$. □

Definition 4.2.2. For a monotone function $e \in \omega^\omega$ that goes to ∞, we define a gauge function e^* by

$$e^*(2^{-k}) = 2^{-e(k)} \text{ for all } k \in \omega.$$

Define the value of $e^*(s)$ for s being not a form of 2^{-k} by linear interpolation.

Lemma 4.2.3. Suppose that $e, h \in \omega^\omega$ nondecreasing satisfy $e(l) \leq \min\{n : l < h(2^n)\}$ for all $l \in \omega$. Then $\mathcal{N}^{e^*} \subseteq \mathcal{J}_h$.

Proof. Let $A \in \mathcal{N}^{e^*}$. Then, for each $n \in \omega$, we can take $\sigma_n \in (2^{<\omega})^\omega$ such that $A \subseteq \bigcup_i [\sigma_n(i)]$ and $\sum_i 2^{-e(|\sigma_n(i)|)} < 2^{-n-2}$. Let $\sigma \in (2^{<\omega})^\omega$ be the enumeration of $\{\sigma_n(i) : n, i \in \omega\}$ in ascending order of length.

It is clear that $A \subseteq [\sigma]_\infty$. So we shall prove that $|\sigma(k)| \geq h(k)$ for all k. Assume that $|\sigma(k)| < h(k)$ for some k. For every $m \leq k$,

$$|\sigma(m)| \leq |\sigma(k)| < h(k) \leq h(2^{n_0}),$$

where $n_0 = \lceil \log_2 k \rceil$. So for $m < k$ we obtain

$$e(|\sigma(m)|) \leq \min\{n : |\sigma(m)| < h(2^n)\} \leq n_0.$$

Then we have

$$\sum_{m<k} 2^{-e(|\sigma(m)|)} \geq \sum_{m<k} 2^{-n_0} = k \cdot 2^{-n_0} \geq 2^{n_0 - 1} 2^{-n_0} = 1/2.$$

On the other hand, by $\sum_i 2^{-e(|\sigma_n(i)|)} < 2^{-n-2}$ for all n, we have $\sum_{k\in\omega} 2^{-e(|\sigma(k)|)} < 1/2$. It is a contradiction. □

Lemma 4.2.4. Let $e, c, h \in \omega^\omega$. Let $\langle I_n : n \in \omega\rangle$ be the interval partition such that $|I_n| = h(n)$. Let $g_{c,h} \colon \omega \to \omega$ be defined by $g_{c,h}(k) = \lfloor \log_2 c(n) \rfloor$ whenever $k \in I_n$. Suppose that $e(g_{c,h}(n)) \geq 2\log_2 n$ for all $n \in \omega$. Then $\mathfrak{v}^\exists_{c,h} \leq \mathrm{non}(\mathcal{N}^{e^*})$ and $\mathrm{cov}(\mathcal{N}^{e^*}) \leq \mathfrak{c}^\exists_{c,h}$.

Proof. This proof is based on [KM22, Lemma 2.4]. We construct a Tukey morphism $(\varphi_-, \varphi_+)\colon \mathbf{Cov}(\mathcal{N}^{e*}) \to \mathbf{wLc}(c,h)$. For each $n \in \omega$, let $\iota_n\colon 2^{\lfloor \log_2 c(n) \rfloor} \to c(n)$ be an injective map. Define φ_- by $\varphi_-(y)(n) = \iota_n(y \upharpoonright \lfloor \log_2 c(n) \rfloor)$. For $S \in S(c,h)$, enumerate the members of S by $S = \{m^S_{n,k} : k \in I_n\}$. For $k \in I_n$, put

$$\sigma_S(k) = \begin{cases} \iota_n^{-1}(m^S_{n,k}) & (\text{if } m^S_{n,k} \in \operatorname{ran} \iota_n) \\ (0)^{\lfloor \log c(n) \rfloor} & (\text{otherwise}). \end{cases}$$

Here $(0)^{\lfloor \log c(n) \rfloor}$ denotes the zero sequence of length $\lfloor \log c(n) \rfloor$. Define $\varphi_+(S) = [\sigma_S]_\infty$.

Then clearly $\varphi_-(y) \in^\infty S \to y \in \varphi_+(S)$. Moreover, we have

$$\begin{aligned} \mathcal{H}^{e^*}_\infty([\sigma_S]_\infty) &= \mathcal{H}^{e^*}_\infty(\bigcap_n \bigcup_{m \geq n} [\sigma_S(m)]) \\ &\leq \mathcal{H}^{e^*}_\infty(\bigcup_{m \geq n} [\sigma_S(m)]) \\ &\leq \sum_{m \geq n} \mathcal{H}^{e^*}_\infty([\sigma_S(m)]) \\ &\leq \sum_{m \geq n} 2^{-e(g_{c,h}(m))} \\ &\leq \sum_{m \geq n} 1/m^2 \to 0 \ (n \to \infty). \end{aligned}$$

So $[\sigma_S]_\infty \in \mathcal{N}^{e^*}$ by Lemma 4.2.1. □

Lemma 4.2.5. Let $e, g \in \omega^\omega$. Suppose that $e(g(i)) \geq 2i$ for all but finitely many i. Then $\mathcal{J}_g \subseteq \mathcal{N}^{e^*}$.

Proof. Let $A \in \mathcal{J}_g$. Then we can take $\sigma \in (2^{<\omega})^\omega$ such that $\mathrm{ht}\sigma = g$ and $A \subseteq [\sigma]_\infty$. Let $\varepsilon > 0$. Now we have

$$(\forall^\infty i)(e(|\sigma(i)|) = e(g(i)) \geq 2i).$$

So

$$(\forall^\infty i)(\ 2^{-e(|\sigma(i)|)} \leq 2^{-2i} \leq 2^{-i}\varepsilon).$$

Modifying the first finitely many terms in σ, we have

$$(\forall i)(2^{-e(|\sigma(i)|)} \leq 2^{-i}\varepsilon).$$

So

$$\sum_i 2^{-e(|\sigma(i)|)} \leq \varepsilon.$$

Thus, $\mathcal{H}^{e^*}_\infty(A) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $A \in \mathcal{N}^{e^*}$ by Lemma 4.2.1. □

Corollary 4.2.6. (1) For every gauge function f, there is an increasing function $g \in \omega^\omega$ such that $\mathcal{I}_g \subseteq \mathcal{N}^f$.

(2) For every increasing function $g \in \omega^\omega$, there is a gauge function f such that $\mathcal{N}^f \subseteq \mathcal{I}_g$. □

Theorem 4.2.7. $\mathcal{I}_{\mathrm{id}} \subsetneq \mathsf{HDZ}$.

Proof. To show $\mathcal{I}_{\mathrm{id}} \subseteq \mathsf{HDZ}$, let $A \in \mathcal{I}_{\mathrm{id}}$. Then we can take $f \gg \mathrm{id}$ and $\sigma \in (2^{<\omega})^\omega$ such that $A \subseteq [\sigma]_\infty$ and $\mathrm{ht}\sigma = f$. Let $s, \varepsilon > 0$. By $f \gg \mathrm{id}$,

$$(\forall^\infty i)\left(f(i) \geq i^2 \geq \frac{i + \log_2(1/\epsilon)}{s}\right).$$

Take $i_0 \in \omega$ such that $(\forall i \geq i_0)(f(i) \geq \frac{i+\log_2(1/\epsilon)}{s})$. Now define $\tau \in (2^{<\omega})^\omega$ as

$$\tau(i) = \begin{cases} (0)^{\lceil (i+\log_2(1/\varepsilon))/s \rceil} & (i < i_0), \\ \sigma(i) & (i \geq i_0). \end{cases}$$

If $x \in 2^\omega$ satisfies $(\exists^\infty i)(\sigma(i) \subseteq x)$ then $(\exists i)(\tau(i) \subseteq x)$. Thus $A \subseteq [\sigma]_\infty \subseteq \bigcup_i [\tau(i)]$.

Also, by $|\tau(i)| \geq (i + \log_2(1/\varepsilon))/s$, $2^{-|\tau(i)|s} \leq \varepsilon/2^i$. Thus we have $\sum_i 2^{-|\tau(i)|s} \leq \varepsilon$. Therefore $A \in \mathsf{HDZ}$.

For $\mathsf{HDZ} \setminus \mathcal{I}_{\mathrm{id}} \neq \varnothing$, take $A = \{x \in 2^\omega : (\forall n \in \omega)(x \restriction I_n \text{ is constant})\}$, where $I_n = [n^2, (n+1)^2)$. To show $A \in \mathsf{HDZ} \setminus \mathcal{I}_{\mathrm{id}}$, first define a tree T as follows:

$$\begin{aligned} T_0 &= \{()\}, \\ T_{n+1} &= \{t^\frown (b)^{|I_n|} : t \in T_n, b \in 2\}, \\ T &= \bigcup_n T_n \downarrow. \end{aligned}$$

Here $T_n \downarrow$ denotes the downward closure of T_n. Clearly, the set of paths through T is A.

Note that

$$A = \bigcap_n \bigcup_{\sigma \in T_n} [\sigma].$$

Let $s > 0$. Then we have

$$\begin{aligned} \mathcal{H}^s_\infty(A) &\leq \mathcal{H}^s_\infty\left(\bigcup_{\sigma \in T_n} [\sigma]\right) \\ &\leq 2^n \cdot 2^{-n^2 s} \\ &\to 0 \text{ (as } n \to \infty). \end{aligned}$$

So we get $\mathcal{H}^s(A) = 0$ by Lemma 4.2.1. Since $s > 0$ is arbitrary, we have $\dim_{\mathrm{H}}(A) = 0$. So $A \in \mathsf{HDZ}$.

To show $A \notin \mathcal{I}_{\mathrm{id}}$, assume that $A \in \mathcal{I}_{\mathrm{id}}$. Then we can take $\sigma \in (2^{<\omega})^\omega$ such that $\mathrm{ht}\sigma \gg \mathrm{id}$ and $A \subseteq [\sigma]_\infty$. We may assume that $\operatorname{ran} \sigma \subseteq T$. Take the natural bijection $\varphi \colon 2^{<\omega} \to T$. Considering $\tau(n) = \varphi^{-1}(\sigma(n))$, we get $2^\omega \subseteq [\tau]_\infty$. Moreover, since φ maps a node whose length is n into a node whose length is n^2,

$$\mathrm{ht}\tau = \sqrt{\mathrm{ht}\sigma} \gg \mathrm{id}.$$

This implies $2^\omega \in \mathcal{I}_{\mathrm{id}}$, contradiction. Thus we get $A \notin \mathcal{I}_{\mathrm{id}}$. ☐

4.3 Additivity number and cofinality of HDZ

Fact 4.3.1 ([Fre08, Theorem 534B]). For every $0 < s < 1$, there is a Tukey isomorphism $\mathbf{Cof}(\mathcal{N}^s) \simeq \mathbf{Cof}(\mathcal{N})$. In particular $\mathrm{cof}(\mathcal{N}^s) = \mathrm{cof}(\mathcal{N})$ and $\mathrm{add}(\mathcal{N}^s) = \mathrm{add}(\mathcal{N})$.

Lemma 4.3.2. Let $\langle I_n : n \in \omega \rangle$ be a sequence of ideals over a set X. Let $I_{n+1} \subseteq I_n$ for every n and $I = \bigcap_n I_n$. Suppose that $\mathrm{add}(I_n) = \kappa$ for every n. Then $\mathrm{add}(I) \geq \kappa$.

Proof. Let $\lambda < \kappa$. Take $\langle X_\alpha : \alpha < \lambda \rangle$ with each $X_\alpha \in I$. Then by the assumption $\bigcup_{\alpha<\lambda} X_\alpha \in I_n$ for all n. Thus we have $\bigcup_{\alpha<\lambda} X_\alpha \in I$. □

Corollary 4.3.3. $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathsf{HDZ})$.

Theorem 4.3.4. Let $\langle I_n : n \in \omega \rangle$ be a decreasing sequence of σ-ideals over a set X and $I = \bigcap_m I_m$. Suppose that for each m, there is a Tukey morphism $(\varphi_m, \psi_m) \colon \mathbf{Cof}(I_m) \to \mathbf{Lc}(\omega, 2^{\mathrm{id}})$. Then there is a Tukey morphism $(\varphi, \psi) \colon \mathbf{Cof}(I) \to \mathbf{Lc}(\omega, 2^{\mathrm{id}})$.

Proof. Fix a bijection $\omega^{<\omega} \to \omega$ and let $\langle a_0, \ldots, a_n \rangle$ denote the image of the n-tuple under this bijection. For $n \in \omega$, let $\mathrm{pr}_n \colon \omega \to \omega$ denote the n-th projection. Put

$$\mathrm{PR}_n \colon S(\omega, 2^{\mathrm{id}}) \to S(\omega, 2^{\mathrm{id}}); S \mapsto (\mathrm{pr}_n \text{“}(S(n)) : n \in \omega).$$

For $A \in I$, define $\varphi(A) \in \omega^\omega$ by

$$\varphi(A)(m) = \langle \varphi_0(A)(m), \ldots, \varphi_m(A)(m) \rangle.$$

For $S \in S(\omega, 2^{\mathrm{id}})$, define $\psi(S) \subseteq X$ by

$$\psi(S) = \bigcup_{m\in\omega} \bigcap_{n \geq m} \psi_n(\mathrm{PR}_n(S)).$$

Since $\varphi_n(\mathrm{PR}_n(S)) \in I_n$, we have $\psi(S) \in I$. Fix $A \in I$ and $S \in S(\omega, 2^{\mathrm{id}})$ such that $\varphi(A) \in^* S$. Then

$$(\forall^\infty i)(\langle \varphi_0(A)(i), \ldots, \varphi_i(A)(i) \rangle \in S(i)).$$

So

$$(\forall^\infty i)(\forall n \leq i)(\varphi_n(A)(i) \in \mathrm{PR}_n(S)(i)).$$

Thus

$$(\forall^\infty n)(\forall i \geq n)(\varphi_n(A)(i) \in \mathrm{PR}_n(S)(i)).$$

Since (φ_n, ψ_n) is Tukey, we have

$$(\forall^\infty n)(A \subseteq \psi_n(\mathrm{PR}_n(S))).$$

Thus, by the definition of ψ, we have

$$A \subseteq \psi(S).$$

□

Corollary 4.3.5. $\mathrm{cof}(\mathsf{HDZ}) \leq \mathrm{cof}(\mathcal{N})$.

4.4 Separating uniformity of $\mathcal{N}$ and $\mathcal{N}^s$

Theorem 4.4.1. (1) For every forcing poset P with Laver property and $s \in (0, 1)$, $P \Vdash 2^\omega \cap V \notin \mathcal{N}^s$.

(2) For every $s \in (0, 1)$, it is consistent with ZFC that $\mathrm{non}(\mathcal{N}^s) < \mathrm{non}(\mathcal{N})$.

(3) For every $0 < s < d$ with $d \in \omega$, it is consistent with ZFC that $\mathrm{non}(\mathcal{N}^s_{\mathbb{R}^d}) < \mathrm{non}(\mathcal{N})$.

Lemma 4.4.2. Let $0 < s < 1$ and $\sigma \in (2^{<\omega})^\omega$. Assume that $\sum_n 2^{-|\sigma(n)|s} \leq 1$ and σ is in ascending order of length. Then $|\sigma(n)| \geq (\log_2 n)/s - C_s$, where $C_s > 0$ is a constant depending only s.

Proof. By the assumption, there are at most 2^{ks} elements of length k in σ. So for all $n \in \omega$, $|\sigma(n)| \geq \alpha(n)$, where

$$\alpha = \langle 0, \underbrace{1, \dots, 1}_{\lceil 2^s \rceil \text{ terms}}, \underbrace{2, \dots, 2}_{\lceil 2^{2s} \rceil \text{ terms}}, \underbrace{3, \dots, 3}_{\lceil 2^{3s} \rceil \text{ terms}}, \dots \rangle.$$

Thus

$$k = \alpha(1 + \lceil 2^s \rceil + \dots + \lceil 2^{(k-1)s} \rceil).$$

So

$$n \geq 1 + \lceil 2^s \rceil + \dots + \lceil 2^{(k-1)s} \rceil \Rightarrow \alpha(n) \geq k.$$

Now for some k_0 we have for all $k \geq k_0$

$$\begin{aligned} 1 + \lceil 2^s \rceil + \dots + \lceil 2^{(k-1)s} \rceil &\leq 1 + 2^s + \dots + 2^{(k-1)s} + k \\ &\leq 1 + 2^s + \dots + 2^{(k-1)s} + 2^{ks} \\ &= (2^{(k+1)s} - 1)/(2^s - 1). \end{aligned}$$

So for $k \geq k_0$,

$$n \geq (2^{(k+1)s} - 1)/(2^s - 1) \Rightarrow \alpha(n) \geq k.$$

That is

$$\log_2((2^s - 1)n + 1)/s - 1 \geq k \Rightarrow \alpha(n) \geq k.$$

So

$$\alpha(n) \geq \log_2((2^s - 1)n + 1)/s - 2,$$

provided that $\log_2((2^s - 1)n + 1)/s - 1 \geq k_0$.

Thus for all but finitely many n we have

$$\begin{aligned} \alpha(n) &\geq \log_2((2^s - 1)n + 1)/s - 2 \\ &\geq \log_2((2^s - 1)n)/s - 2 \\ &= \log_2 n/s + \log_2(2^s - 1)/s - 2. \end{aligned}$$

So putting $C_s = 2 - \log_2(2^s - 1)/s$ gives

$$\alpha(n) \geq \log_2 n/s - C_s.$$

By adjusting C_s, we can make the above inequality hold for all n. $\square$

Lemma 4.4.3. Let $A \subseteq 2^\omega$ and $s > 0$. If $\mathcal{H}^s(A) = 0$, then there is $\sigma \in (2^{<\omega})^\omega$ such that

$$\sum_n 2^{-|\sigma(n)| \cdot s} \leq 1 \;\&\; A \subseteq [\sigma]_\infty.$$

Proof. By $\mathcal{H}^s_\infty(A) = 0$, for each $n \in \omega$, we can take $\langle C^n_m : m \in \omega \rangle$ such that $\sum_m \operatorname{diam}(C^n_m)^s \leq 2^{-(n+1)}$ and $A \subseteq \bigcup_{m\in\omega} C^n_m$. Expand each C^n_m to a basic open set $[\sigma^n_m]$ so that its diameter does not change. Let $\sigma \in (2^{<\omega})^\omega$ be an enumeration of $\langle \sigma^n_m : n, m \in \omega \rangle$. Then we have $\sum_n 2^{-|\sigma(n)|\cdot s} \leq 1$ and $A \subseteq [\sigma]_\infty$. $\square$

Lemma 4.4.4. For $0 < s < 1$, there is a Borel relational system $\mathbf{Cov}'(\mathcal{N}^s)$ that is equivalent to $\mathbf{Cov}(\mathcal{N}^s)$.

Proof. Define H and $\lhd$ as follows

$$\begin{aligned} H =& \{\sigma \in (2^{<\omega})^{\omega\times\omega} : (\forall m \in \omega)(\sum_n 2^{-\sigma(n,m)\cdot s} \leq 1/(m+1))\}, \\ x \lhd \sigma \iff& (\forall m)(x \in \bigcup_{n\in\omega} [\sigma(n,m)]). \end{aligned}$$

Then $\mathbf{Cov}'(\mathcal{N}^s) = (2^\omega, H, \lhd)$ suffices. $\square$

Proof of Theorem 4.4.1. To show item (1), fix $\dot\sigma$ and $p \in \mathbb{P}$ such that $p \Vdash \dot\sigma \in (2^{<\omega})^\omega$ and $p \Vdash \sum_n 2^{-s|\dot\sigma(n)|} \leq 1/2$. Fix $q \leq p$. Put $\beta(n) = \lfloor (\log_2 n)/s - C_s \rfloor$ where C_s is the constant from Lemma 4.4.2. Define $\dot\tau$ so that $p \Vdash \dot\tau(n) = \dot\sigma(n) \restriction \beta(n)$.

By the Laver property, we can take $r \leq q$ and $S \in \prod_{n\in\omega} [{}^{\beta(n)}2]^{\leq n^{(1-s)/2}}$ such that $r \Vdash (\forall^\infty n)(\dot\tau(n) \in S(n))$. Let X be the Borel code of $\bigcap_{k\in\omega} \bigcup_{n\geq k} \bigcup\{[t] : t \in S(n)\}$. Then we have $r \Vdash [\dot\tau]_\infty \subseteq \hat{X}$.

Now we have the following:

$$\begin{aligned} \mathcal{H}^{(1+s)/2}_\infty(\hat{X}) &\leq \mathcal{H}^{(1+s)/2}_\infty(\bigcup_{n\geq k} \bigcup\{[t] : t \in S(n)\}) \\ &\leq \sum_{n\geq k} n^{(1-s)/2}(2^{-\beta(n)})^{(1+s)/2} \\ &\leq 2^{(C_s+1)(1+s)/2} \sum_{n\geq k} n^{(1-s)/2} n^{-(1+s)/(2s)} \\ &= 2^{(C_s+1)(1+s)/2} \sum_{n\geq k} n^{-(1/2)(s+1/s)} \to 0 \text{ (as } k \to \infty). \end{aligned}$$

We used $(1/2)(s + 1/s) > 1$ in the last equation. Thus $\mathcal{H}^{(1+s)/2}(\hat{X}) = 0$. Since $(1 + s)/2 < 1$ and $\dim_{\mathrm{H}}(2^\omega) = 1$, we have $\hat{X} \neq 2^\omega$.

Then we can take $x \in 2^\omega \smallsetminus \hat{X}$ in V. Then by absoluteness, we also have $r \Vdash x \in 2^\omega \smallsetminus \hat{X}$. By $r \Vdash [\dot\sigma]_\infty \subseteq [\dot\tau]_\infty \subseteq \hat{X}$, we have $r \Vdash x \notin [\dot\sigma]_\infty$. Therefore we have $\Vdash (\forall \sigma \in (2^{<\omega})^\omega)(\sum_n 2^{-|\sigma(n)|s} \leq 1 \Rightarrow 2^\omega \cap V \not\subseteq [\dot\sigma]_\infty)$. So by Lemma 4.4.3, we obtain $\Vdash \mathcal{H}^s(2^\omega \cap V) > 0$.

For item (2), consider ω_2-step countable support iteration of Mathias forcing over a model of CH. In this model, by the item (1) and Lemma 4.4.4, $\operatorname{non}(\mathcal{N}^s) = \aleph_1$ whereas $\operatorname{non}(\mathcal{N}) = \aleph_2$.

For item (3), use item (2) and Proposition 4.1.4. In detail, let $0 < s < d$ and put $s' = s(1+\varepsilon)/d < 1$ for some $\varepsilon > 0$. By Proposition 4.1.3, Proposition 4.1.4 and Proposition 4.1.8, there is a $(1 + \varepsilon)/d$-co-Hölder map $f : 2^\omega \to [0, 1]^d$. Take $A \subseteq 2^\omega$ such that $|A| = \operatorname{non}(\mathcal{N}^s)$ and $\mathcal{H}^{s'}(A) > 0$. Then $\mathcal{H}^s(f(A)) \geq C \cdot \mathcal{H}^{s'}(A) > 0$ for some constant $C > 0$ by Proposition 4.1.2. Now we have $|f(A)| \leq |A| = \mathit{non}(\mathcal{N}^s)$, so $\operatorname{non}(\mathcal{N}^s_{[0,1]^d}) \leq \operatorname{non}(\mathcal{N}^{s'})$. Thus in the model of (2), we have $\operatorname{non}(\mathcal{N}^s_{\mathbb{R}^d}) < \operatorname{non}(\mathcal{N})$. $\square$

Remark 4.4.5. The consistency of Theorem 4.4.1 (2) was already proved in [SS05]. But the forcing posets are simpler in our work than in their work.

4.5 Many different uniformity numbers of Hausdorff measure 0 ideals

In this section, we prove the following theorem.

Theorem 4.5.1. It is consistent with ZFC that there are $\aleph_1$ many cardinals of the form $\mathrm{non}(\mathcal{N}^f)$ below the continuum.

We modify the proof that there are consistently many different uniformity numbers of Yorioka ideals from [KM22].

Definition 4.5.2. (1) For $c, h \in \omega^\omega$, define $g_{c,h} \in \omega^\omega$ by

$$g_{c,h}(k) = \lfloor \log_2 c(n) \rfloor \text{ (whenever } k \in J_n)$$

where $(J_n)_{n\in\omega}$ is the interval partition with $|J_n| = h(n)$ for all $n \in \omega$.

(2) For $b, g \in \omega^\omega$, define $f_{b,g} \in \omega^\omega$ by

$$f_{b,g}(k) = \sum_{l \leq n} \lceil \log_2 b(l) \rceil \text{ (whenever } k \in I_n)$$

where $(I_n)_{n\in\omega}$ is the interval partition with $|I_n| = g(n)$ for all $n \in \omega$.

(3) For $f \in \omega^\omega$ increasing, define $e_f \in \omega^\omega$ by

$$e_f(k) = \min\{n \in \omega : k < f(2^n)\}.$$

(4) For $c, h \in \omega^\omega$ define $c^{\nabla h} \in \omega^\omega$ by

$$c^{\nabla h}(n) = |[c(n)]^{\leq h(n)}|.$$

Definition 4.5.3 ([KM22, Definition 4.1])**.** Two functions $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$ of natural numbers ≥ 2 are called *bounding sequences* if

(i) $n_k^- \cdot n_k^+ < n_{k+1}^-$ for all $k \in \omega$, and

(ii) $\lim_{k\to\infty} \log_{n_k^-} n_k^+ = \infty$.

Given bounding sequences $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$, a family $\mathcal{F} = \{(a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) : \alpha \in A\}$ of tuples of increasing functions in ω^ω is called *suitable* with respect to $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$ if it satisfies the following properties for all $\alpha \in A$:

(S1) For all $k \in \omega$, we have $a_\alpha(k), d_\alpha(k), b_\alpha(k), g_\alpha(k), b_\alpha^{\nabla g_\alpha}(k), \frac{b_\alpha(k)}{g_\alpha(k)}, h_\alpha(k), c_\alpha^{\nabla h_\alpha}(k) \in [n_k^-, n_k^+]$.

(S2) $h_\alpha < c_\alpha$ and $\limsup_{k\to\infty} \frac{1}{d_\alpha(k)} \log_{d_\alpha(k)}(h_\alpha(k)+1) = \infty$.

(S3) $b_\alpha/g_\alpha \geq d_\alpha$.

(S4) $a_\alpha \geq b_\alpha^{\nabla g_\alpha}$.

(S5) There is some $l > 0$ such that $f_{b_\alpha, g_\alpha} \leq^* f_\alpha \circ \mathrm{pow}_l$.

(S6) $f_\alpha \ll g_{c_\alpha,h_\alpha}$.

(S7) For all $\beta \in A$ with $\beta \neq \alpha$,

$$\lim_{k\to\infty} \min\left\{\frac{c_\beta^{\nabla h_\beta}(k)}{d_\alpha(k)}, \frac{a_\alpha(k)}{d_\beta(k)}\right\} = 0.$$

Fact 4.5.4 ([KM22, Section 4 and 5])**.** (1) There are bounding sequences and there is suitable family $\mathcal{F}$ of continuum size with respect to them.

(2) Assume CH and let $(\kappa_\alpha : \alpha \in A)$ be a sequence of infinite cardinals such that $|A| \leq \aleph_1$ and $\kappa_\alpha^\omega = \kappa_\alpha$ for all $\alpha \in A$. Given a family $\mathcal{F} = \{(a_\alpha, d_\alpha, b_\alpha, g_\alpha, f_\alpha, c_\alpha, h_\alpha) : \alpha \in A\}$ satisfying (S1) and (S7) with respect to some bounding sequences, there is a forcing poset that preserves all cardinals and forces

$$\mathfrak{c}^\forall_{a_\alpha,d_\alpha} \leq \kappa_\alpha \leq \mathfrak{v}^\exists_{c_\alpha,h_\alpha}.$$

for all $\alpha \in A$. If the family $\mathcal{F}$ is suitable, then

$$\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{I}_{f_\alpha}) \leq \mathfrak{v}^\exists_{b_\alpha,g_\alpha} \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}$$

is a ZFC theorem. Thus the forcing poset forces

$$\mathfrak{v}^\exists_{c_\alpha,h_\alpha} = \mathrm{non}(\mathcal{I}_{f_\alpha}) = \mathfrak{v}^\exists_{b_\alpha,g_\alpha} = \mathfrak{c}^\forall_{a_\alpha,d_\alpha} = \kappa_\alpha$$

for all $\alpha \in A$.

Definition 4.5.5. Given bounding sequences $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$, a family $\mathcal{F} = \{(a_\alpha, d_\alpha, b_\alpha, g_\alpha, c_\alpha, h_\alpha, e_\alpha, u_\alpha) : \alpha \in A\}$ of tuples of increasing functions in ω^ω is called *modified suitable* with respect to $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$ if it satisfies (S1), (S2), (S3), (S4), (S7) and the following (MS1), (MS2) and (MS3) for all $\alpha \in A$:

(MS1) $e_\alpha = e_{u_\alpha}$.

(MS2) $e_\alpha(g_{c_\alpha,h_\alpha}(k)) \geq 2\log_2 k$ for all $k \in \omega$.

(MS3) $f_{b_\alpha,g_\alpha} \leq u_\alpha$.

Proposition 4.5.6. For a modified suitable family $\mathcal{F} = \{(a_\alpha, d_\alpha, b_\alpha, g_\alpha, c_\alpha, h_\alpha, e_\alpha, u_\alpha) : \alpha \in A\}$, we have

$$\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{N}^{e^*_\alpha}) \leq \mathrm{non}(\mathcal{J}_{u_\alpha}) \leq \mathfrak{v}^\exists_{b_\alpha,g_\alpha} \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}.$$

Proof. $\mathfrak{v}^\exists_{c_\alpha,h_\alpha} \leq \mathrm{non}(\mathcal{N}^{e^*_\alpha})$ follows from (MS2) and Lemma 4.2.4. $\mathrm{non}(\mathcal{N}^{e^*_\alpha}) \leq \mathrm{non}(\mathcal{J}_{u_\alpha})$ follows from (MS1) and Lemma 4.2.3. $\mathrm{non}(\mathcal{J}_{u_\alpha}) \leq \mathfrak{v}^\exists_{b_\alpha,g_\alpha}$ follows from (MS3) and [KM22, Lemma 2.5]. $\mathfrak{v}^\exists_{b_\alpha,g_\alpha} \leq \mathfrak{c}^\forall_{a_\alpha,d_\alpha}$ follows from (S3), (S4) and [KM22, Lemma 2.6]. □

Proposition 4.5.7. There are bounding sequences and there is a modified suitable family $\mathcal{F}$ of continuum size with respect to them.

Proof. First, we build bounding sequences $(n_k^-)_{k\in\omega}, (n_k^+)_{k\in\omega}$ and a modified suitable family $\mathcal{F} = \{(a, d, b, g, c, h, e, u)\}$ of size 1 by recursion. Let $n_0^- = 2$ and $d(0) = 3$. Let $\langle I_n : n \in \omega\rangle$ and $\langle J_n : n \in \omega\rangle$ be interval partitions with $|I_n| = g(n)$ and $|J_n| = h(n)$. Define the component of $\mathcal{F}$ in the following order:

(1) $h(k) = d(k)^{(k+1)d(k)}$,

(2) $g(k) = \max\{(\max J_k)^2 - \min I_k + 1, h(k) + 1\}$,

(3) $b(k) = 2^{g(k)+d(k)}$,

(4) $u(j) = \sum_{l \leq k} \log_2 b(l) + j - \min I_k$ for $j \in I_k$,

(5) $c(k) = 2^{u((\max J_k)^2)-1}$,

(6) $a(k) = \max\{c^{\nabla h}(k), b^{\nabla g}(k)\} + 1$,

(7) $n_k^+ = a(k)$,

(8) $n_{k+1}^- = n_k^- \cdot n_k^+ + 1$ and

(9) $d(k+1) = n_{k+1}^- + 1$.

Item (1), (3), and (6) ensures (S2), (S3), and (S4) respectively.

Item (4) ensures (MS3) since

$$\begin{aligned} u(j) &= \sum_{l \leq k} \log_2 b(l) + j - \min I_k \\ &\geq \sum_{l \leq k} \log_2 b(l) \\ &= f_{b,g}(j) \end{aligned}$$

for $j \in I_k$. Moreover, this definition ensures u is strictly increasing since

$$\begin{aligned} u(\max I_{k-1}) &= \sum_{l \leq k-1} \log_2 b(l) + \max I_{k-1} - \min I_{k-1} \\ &= \sum_{l \leq k-1} \log_2 b(l) + g(k-1) - 1 \\ &< \sum_{l \leq k} \log_2 b(l) \\ &= u(\min I_k). \end{aligned}$$

When we are done defining $(n_k^-)_{k \in \omega}, (n_k^+)_{k \in \omega}, a, d, b, g, c, h$ and u, we define e by

$$e = e_u.$$

This ensures (MS1).

Item (5) ensures (MS2) since

$$g_{c,h}(k) = u((\max J_k)^2) - 1 \geq u(k^2) - 1$$

for $k \in J_n$ and

$$\begin{aligned} e_u(g_{c,h}(k)) &= \min\{n : g_{c,h}(k) < u(2^n)\} \\ &\geq \min\{n : u(k^2) - 1 < u(2^n)\} \\ &= \lceil \log_2 k^2 \rceil \\ &\geq 2 \log_2 k. \end{aligned}$$

Item (2) ensures that in Item (5) we will not access u with an invalid index. In fact, from Item (2) we obtain

$$g(k) \geq (\max J_k)^2 - \min I_k + 1.$$

So we have

$$(\max J_k)^2 \leq \min I_k + g(k) - 1 = \max I_k.$$

Thus when we are in (5), we already defined $u((\max J_k)^2)$.

The above construction ensures

$$n_k^- < d(k) < h(k) < g(k) < b(k) < b^{\nabla g}(k) < a(k) = n_k^+,$$

$$n_k^- < d(k) < b(k)/g(k) < b(k) < n_k^+$$

and

$$n_k^- < d(k) < h(k) < c(k) < c^{\nabla h}(k) < a(k) = n_k^+$$

So (S1) holds.

Now we shall show how to construct a modified suitable family $\mathcal{F} = \{(a_\alpha, d_\alpha, b_\alpha, g_\alpha, c_\alpha, h_\alpha, e_\alpha, u_\alpha) : \alpha \in 2^\omega\}$ of size continuum. We construct approximations $\langle (a_t, d_t, b_t, g_t, c_t, h_t, u_t) : t \in 2^{<\omega} \rangle$ and then put for $\alpha \in 2^\omega$, $a_\alpha = \bigcup_{n\in\omega} a_{\alpha\restriction n}$, etc.

Let $\bar{0} = \langle 0, 0, 0, \dots \rangle$, $\bar{1} = \langle 1, 1, 1, \dots \rangle$. Let $\lhd$ denote the lexicographical order of 2^ω and 2^n for $n \in \omega$. By recursion on $n \in \omega$ we define $\langle (a_t, d_t, b_t, g_t, c_t, h_t, u_t) : t \in 2^n \rangle$.

(1) Let $d_{\bar{0}\restriction(n+1)}(n) > d_{\bar{0}\restriction(n)}(n-1) \cdot a_{\bar{1}\restriction(n)}(n-1) + 2$.

(2) When $d_t(n)$ is defined, put $d_{t^+}(n) = (n+1)a_t(n)$, where t^+ is the successor of t in $\lhd$.

(3) Define $h_{t^+}(n), g_{t^+}(n), \dots, a_{t^+}(n)$ as in the construction of the modified suitable family of size 1.

Put $n_k^- = d_{\bar{0}}(k) - 1$ and $n_k^+ = a_{\bar{1}}(k)$. And put $e_\alpha = e_{u_\alpha}$ for $\alpha \in 2^\omega$.

We finished the construction and have to check (S7). It suffices that we prove for $\alpha \lhd \beta$, $\lim_{k\to\infty} \frac{a_\alpha(k)}{d_\beta(k)} = 0$. Let n be the minimum number such that $\alpha(n) < \beta(n)$. Then by the definition of d_β, we have $d_\beta(k) \geq (k+1)a_\alpha(k)$ for any $k \geq n$. Thus $\frac{a_\alpha(k)}{d_\beta(k)} \to 0$ (as $k \to 0$). □

By Fact 4.5.4 (2), Propositon 4.5.6 and 4.5.7, we have Theorem 4.5.1.

4.6 Many different covering numbers of Hausdorff measure 0 ideals

The following fact was proved by Kamo and Osuga in [OK14, Section 3].

Fact 4.6.1. Let δ be an ordinal and $\langle \lambda_\alpha : \alpha < \delta \rangle$ be a strictly increasing sequence of regular uncountable cardinals. Let $\kappa \geq \delta$ be a cardinal such that $\kappa = \kappa^{<\lambda_\alpha}$ for all $\alpha < \delta$. Let $\langle b_\alpha, c_\alpha : \alpha < \delta \rangle$ be a sequence of pairs of reals in ω^ω such that $b_\alpha >^* c_\beta^H \cdot \mathrm{id}$ for all $\beta < \alpha < \delta$ and $b_\alpha >^* 2^{\mathrm{id}}$ for all $\alpha < \delta$, where $H = \langle n^{n^2} : n \in \omega \rangle$. Then there is a ccc forcing poset $\mathbb{P}$ such that

$$\mathbb{P} \Vdash ((\forall \alpha < \delta)(\mathfrak{c}^{\exists}_{c_\alpha, H} \leq \lambda_\alpha \leq \mathfrak{c}^{\exists}_{b_\alpha, 1}) \,\&\, \mathfrak{c} = \kappa).$$

Theorem 4.6.2. It is consistent with ZFC that there are $\aleph_1$ many cardinals of the form $\mathrm{cov}(\mathcal{N}^f)$ below the continuum.

Proof. Assume GCH. Put $\delta = \omega_1$. Put $\lambda_\alpha = \aleph_{\alpha+1}$ for $\alpha < \omega_1$. Put $\kappa = \aleph_{\omega_1+1}$. We define $\langle b_\alpha, g_\alpha, e_\alpha, c_\alpha : \alpha < \omega_1\rangle$ recursively so that

(1) $b_\alpha >^* 2^{\mathrm{id}}$ for all $\alpha < \omega_1$,

(2) $b_\alpha >^* c_\beta^H \cdot \mathrm{id}$ for all $\beta < \alpha < \omega_1$,

(3) $g_\alpha(n) \geq \sum_{i \leq n} \log_2 b_\alpha(i)$,

(4) $e_\alpha(n) = \min\{m \in \omega : n < g_\alpha(2^m)\}$ and

(5) c_α satisfies $e_\alpha(g_{c_\alpha,H}(n)) \geq 2\log_2(n)$ for all $n \in \omega$ and $\alpha < \omega_1$.

Then, the assumption of Fact 4.6.1 holds. So we can take a ccc forcing poset $\mathbb{P}$ such that

$$\mathbb{P} \Vdash (\forall \alpha < \omega_1)(\mathfrak{c}^\exists_{c_\alpha,H} \leq \lambda_\alpha \leq \mathfrak{c}^\exists_{b_\alpha,1}).$$

But by item (3) above and [OK14, Lemma 1], we have $\mathfrak{c}^\exists_{b_\alpha,1} \leq \mathrm{cov}(\mathcal{J}_{g_\alpha})$. And item (4) and Lemma 4.2.3 gives $\mathrm{cov}(\mathcal{J}_{g_\alpha}) \leq \mathrm{cov}(\mathcal{N}^{e^*_\alpha})$. Item (5) and Lemma 4.2.4 gives $\mathrm{cov}(\mathcal{N}^{e^*_\alpha}) \leq \mathfrak{c}^\exists_{c_\alpha,H}$.

Therefore we have

$$\mathbb{P} \Vdash (\forall \alpha < \omega_1)(\mathfrak{c}^\exists_{b_\alpha,1} = \mathrm{cov}(\mathcal{J}_{g_\alpha}) = \mathrm{cov}(\mathcal{N}^{e^*_\alpha}) = \mathfrak{c}^\exists_{c_\alpha,H} = \lambda_\alpha).$$

Especially we have

$$\mathbb{P} \Vdash (\forall \alpha < \omega_1)(\mathrm{cov}(\mathcal{N}^{e^*_\alpha}) = \lambda_\alpha).$$

□

4.7 Goldstern's principle of Hausdorff measures

We consider the generalization of Goldstern's principle, which was considered in the previous chapter, to Hausdorff measures.

Let $\mathcal{I}$ be an ideal on some Polish space X. We say Goldstern's principle for a pointclass Γ with respect to $\mathcal{I}$ holds if for every $A \subseteq \omega^\omega \times X$ satisfying the monotonicity condition, the condition $A \in \Gamma$ and the condition $A_x \in \mathcal{I}$ for every $x \in \omega^\omega$, we have $\bigcup_{x\in\omega^\omega} A_x \in \mathcal{I}$.

For a gauge function f, let $I_{\sigma f}$ be the ideal of all set of reals of σ-finite f-Hausdorff measure. Recall that $\mathbb{P}_{\mathcal{I}}$ stands for the idealized forcing for an ideal $\mathcal{I}$.

A gauge function f is called a doubling gauge function if there is $r > 0$ such that for every $x > 0$ we have $f(2x) < rf(x)$.

Theorem 4.7.1. Let f be a continuous doubling gauge function and X be a compact metric space. Then Goldstern's principle for $\mathbf{\Sigma}^1_1$ sets with respect to $\mathcal{N}^f_X$ holds.

Proof. Let $A \subseteq \omega^\omega \times X$ be a $\mathbf{\Sigma}^1_1$ set satisfying the monotonicity condition and assume A_x is of f-Hausdorff measure 0 for each $x \in \omega^\omega$. Assume also that $\bigcup_{x\in\omega^\omega} A_x$ is not of f-Hausdorff measure 0.

We divide the argument into two cases.

Firstly, we consider the case when $\bigcup_{x\in\omega^\omega} A_x$ is of non-σ-finite f-Hausdorff measure. Since $\bigcup_{x\in\omega^\omega} A_x$ is $\mathbf{\Sigma}^1_1$, we can take a Borel set B such that $B \subseteq \bigcup_{x\in\omega^\omega} A_x$ and B is of non-σ-finite f-Hausdorff measure. The existence of such a Borel set B is ensured by the assumption f is continuous and X is compact and the theorem by M. Sion and D. Sjerve [SS62, Theorem 6.6].

Let G be a $(V, \mathbb{P}_{I_{\sigma f}})$-generic filter such that $B \in G$. And let g be the corresponding generic real to G. By genericity, we have $g \notin A_x$ for each $x \in \omega^\omega \cap V$. Thus we have $g \notin \bigcup_{x\in\omega^\omega\cap V} A_x$. Since the monotonicity condition holds, which is absolute between V and $V[G]$, and since $\mathbb{P}_{I_{\sigma f}}$ is ω^ω-bounding, which is the Theorem by J. Zapletal [Zap08, Corollary 4.4.2], we have $g \notin \bigcup_{x\in\omega^\omega} A_x$. It contradicts the fact $g \in B \subseteq \bigcup_{x\in\omega^\omega} A_x$.

Secondly, we consider the case when $\bigcup_{x\in\omega^\omega} A_x$ is of σ-finite f-Hausdorff measure. Let $\langle B_n : n \in \omega\rangle$ be a sequence of Borel sets such that $\bigcup_{x\in\omega^\omega} A_x \subseteq \bigcup_n B_n$ and each B_n is of finite positive f-Hausdorff measure. For each n, consider the set $\bigcup_{x\in\omega^\omega}(A_x \cap B_n)$. Since B_n is of finite positive f-Hausdorff measure, the restriction of f-Hausdorff measure into the Borel subsets in B_n is measure isomorphic to the Lebesgue measure by using measure isomorphism theorem. Therefore, using $\mathsf{GP}(\mathbf{\Sigma}^1_1)$, we conclude $\bigcup_{x\in\omega^\omega}(A_x \cap B_n)$ is of f-Hausdorff measure zero. So taking the union over $n \in \omega$, we deduce that $\bigcup_{x\in\omega^\omega} A_x$ is of f-Hausdorff measure zero. □

4.8 Laver forcing preserves Hausdorff outer measures

In order to prove the consistency of Goldstern's principle of Hausdorff measures for the pointclass all, we show that the Laver forcing preserves Hausdorff measures. However, this alone does not achieve the objective. To achieve it, we must show that countable support iterations of the Laver forcing also preserve Hausdorff measures. Unfortunately, we could not prove this.

Recall that $\mathbb{L}$ denotes the Laver forcing.

Definition 4.8.1. An outer measure μ^* on a space X satisfies the *increasing sets lemma* if $\mu^*(\bigcup_n A_n) \leq \sup \mu^*(A_n)$ for every increasing sequence $\langle A_n : n \in \omega\rangle$ of subsets of X.

Theorem 4.8.2 ([Dav70])**.** If X is a compact metric space, f is a continuous gauge function and $\delta > 0$, then $\mathcal{H}^f_\delta$ satisfies the increasing sets lemma.

Definition 4.8.3. For $T \in \mathbb{L}$ and $t \in T$, let T_t denote the set $\{s \in T : t \subseteq s \vee s \subseteq t\}$. For $T \in \mathbb{L}$ and $\tau \in \omega^{<\omega}$, let $T(\tau)$ denote the image of τ under the canonical isomorphism from $\omega^{<\omega}$ into T. For $T \in \mathbb{L}$ and $\tau \in \omega^{<\omega}$, let $T\langle\tau\rangle = T_{T(\tau)}$.

For $S, T \in \mathbb{L}$ and $n \in \omega$, the relation $S \leq_n T$ holds if $S(\tau) = T(\tau)$ for every $\tau \in \omega^n$.

For open sets $D \subseteq \mathbb{L}$, let

$$\tilde{D} = \{T \in \mathbb{L} : \forall S \leq_0 T\ S \notin D\} \cup D, \text{ and}$$
$$D^* = \{T \in \mathbb{L} : \exists n\ \forall \tau \in \omega^{<\omega}\ |\tau| > n \Rightarrow T\langle\tau\rangle \in \tilde{D}\}.$$

Lemma 4.8.4. (1) If $D \subseteq \mathbb{L}$ is open and nonempty below $S \leq T \in D^*$, then there is $s \in S$ such that $T_s \in D$.

(2) If $D \subseteq \mathbb{L}$ is open, then for every $i \in \omega$ and every $T \in \mathbb{L}$, there is $S \leq_i T$ such that $S \in D^*$.

(3) If $D_n \subseteq \mathbb{L}$ $(n \in \omega)$ are open, then for every $i \in \omega$ and every $T \in \mathbb{L}$, there is $S \leq_i T$ such that $S \in \bigcap_n D^*_n$.

Proof. See [Paw96]. $\square$

In this section, we prove that one step Laver forcing preserves Hausdorff outer measures. The following lemma is crucial.

Lemma 4.8.5. Let μ^* be an outer measure on 2^ω such that the increasing sets lemma holds for μ^*. Let $a > 0$ be a real number. Let $A_\sigma \subseteq 2^\omega$ ($\sigma \in \omega^{<\omega}$) satisfy the following conditions:

(1) For every $\sigma \in \omega^{<\omega}$, $\mu^*(A_\sigma) \leq a$.

(2) For every $\sigma \in \omega^{<\omega}$, $A_\sigma \subseteq \liminf_n A_{\sigma^\frown n}$.

Then we have $\mu^*(\bigcap_{T\in\mathbb{L}} \bigcup_{\sigma\in T} A_\sigma) \leq a$.

Proof. Recursively we define the sequence $\langle A_\sigma^\alpha : \alpha < \omega_1, \sigma \in \omega^{<\omega}\rangle$ as follows:

$$\begin{aligned} A_\sigma^0 &= A_\sigma, \\ A_\sigma^{\alpha+1} &= \liminf_{n\in\omega} A_{\sigma^\frown n}^\alpha, \\ A_\sigma^\lambda &= \bigcup_{\alpha<\lambda} A_\sigma^\alpha \quad \text{(for limit } \lambda). \end{aligned}$$

Note that, for every $\sigma \in \omega^{<\omega}$, $\langle A_\sigma^\alpha : \alpha < \omega_1\rangle$ is an increasing sequence and each member has outer measure $\leq a$ using induction on α and the increasing sets lemma. Therefore, there is $\alpha_\sigma < \omega_1$ such that $\mu^*(A_\sigma^{\beta+1} \smallsetminus A_\sigma^\beta) = 0$ for every $\beta \geq \alpha_\sigma$. Put $\alpha = \sup_\sigma \alpha_\sigma$. Then we have $\mu^*(A_\varnothing^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \smallsetminus A_\sigma^\alpha)) \leq a$.

We now claim that $\bigcap_{T\in\mathbb{L}} \bigcup_{\sigma\in T} A_\sigma \subseteq A_\varnothing^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \smallsetminus A_\sigma^\alpha)$. To show it, let $x \notin A_\varnothing^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \smallsetminus A_\sigma^\alpha)$. We build a sequence of trees $\langle T_i : i \in \omega\rangle$ such that each T_i has height i and for each maximal element σ of T_i, we have $x \notin A_\sigma^{\alpha+1}$. Let $T_0 = \{\varnothing\}$. By $x \notin A_\varnothing^{\alpha+1}$, the base case of induction works. Suppose that T_i has been constructed. For every maximal element σ of T_i, we have $x \notin A_\sigma^{\alpha+1}$. By the definition of $A_\sigma^{\alpha+1}$, the set $X_\sigma := \{n : x \notin A_{\sigma^\frown n}^\alpha\}$ is infinite. Let $T_{i+1} = T_i \cup \{\sigma^\frown n : \sigma \text{ maximal element of } T_i, n \in X_\sigma\}$. If $\sigma^\frown n$ is a maximal element of T_{i+1}, then by $x \notin A_{\sigma^\frown n}^{\alpha+1} \smallsetminus A_{\sigma^\frown n}^\alpha$, we have $x \notin A_{\sigma^\frown n}^{\alpha+1}$. Therefore the induction hypothesis continues to hold. Finally we put $T = \bigcup_{i\in\omega} T_i$. Then we have $x \notin \bigcup_{\sigma\in T} A_\sigma^{\alpha+1}$. In particular, we have $x \notin \bigcup_{\sigma\in T} A_\sigma$. $\square$

Definition 4.8.6. Let P be a forcing notion. Let $\dot{U}$ be a P-name such that $P \Vdash \dot{U}$ is a finite subset of $2^{<\omega}$. We define

$$\mathrm{dec}(\dot{U}) = \{p \in \mathbb{L} : p \text{ decides } \dot{U}\}.$$

Also for $p \in \mathrm{dec}(\dot{U})$, let $\dot{U}(p)$ be the decided value of $\dot{U}$ by p. For $p \notin \mathrm{dec}(\dot{U})$, let $\dot{U}(p) = \varnothing$. Let

$$\mu_\delta^f(\dot{U}) = \sup\left\{\sum_{s\in\dot{U}(p)} f(\mathrm{diam}([s])) : p \in \mathrm{dec}(\dot{U})\right\}$$

provided that $P \Vdash (\forall s \in \dot{U})(\mathrm{diam}([s]) \leq \delta)$. Otherwise let $\mu_\delta^f(\dot{U}) = \infty$.

Lemma 4.8.7. If $R \leq T$, $R \in \mathrm{dec}(\dot{U})$ and $T \in \mathrm{dec}(\dot{U})^*$, then there is $s \in R$ such that $\dot{U}(T_s) = \dot{U}(R)$.

Proof. Since $\mathrm{dec}(\dot{U})$ is nonempty below R, by Lemma 4.8.4 (1), we have $T_s \in \mathrm{dec}(\dot{U})$ for some $s \in R$. T_s is in $\mathrm{dec}(\dot{U})$, so $\dot{U}(R) = \dot{U}(R_s) = \dot{U}(T_s)$. $\square$

Lemma 4.8.8. Let $\dot{U}_n$ $(n \in \omega)$ be $\mathbb{L}$-names for finite subsets of $2^{<\omega}$. If $T \in \bigcap_{n\in\omega} \operatorname{dec}(\dot{U}_n)^*$, then

$$\mathcal{H}^f_\delta(\{x \in 2^\omega : T \Vdash x \in \bigcup_n \bigcup_{s\in\dot{U}_n} [s]\}) \le \sum_{n\in\omega} \mu^f_\delta(\dot{U}_n).$$

Proof. We may assume that $\mathbb{L} \Vdash (\forall s \in \dot{U})(\operatorname{diam}([s]) \le \delta)$. We have the following equivalence:

$$\begin{aligned} &T \Vdash x \in \bigcup_n \bigcup_{s\in\dot{U}_n} [s] \\ &\iff (\forall S \le T)(\exists R \le S)(\exists n)(\exists s \in \dot{U}_n(R)) x \in [s] \\ &\iff (\forall S \le T)(\exists t \in S)(\exists n)(\exists s \in \dot{U}_n(T_t)) x \in [s] \\ &\iff x \in \bigcap_{S\le T} \bigcup_{t\in S} \bigcup_n \bigcup_{s\in\dot{U}_n(T_t)} [s]. \end{aligned}$$

Here, we used Lemma 4.8.7 for the second equivalence.

On the other hand, we have $\mathcal{H}^f_\delta(\bigcup_n \bigcup_{s\in\dot{U}_n(T_t)}[s]) \le \sum_n \mu^f_\delta(\dot{U}_n)$ for every $t \in T$. Therefore we obtain the conclusion by Lemma 4.8.5. □

Definition 4.8.9. Let P be a forcing notion. Let f be a gauge function, $\delta > 0$ and $\varepsilon > 0$. Let

$$\begin{aligned} I^{P,f}_{\varepsilon,\delta} = \{\langle \dot{U}_n : n \in \omega\rangle : P \Vdash \text{“}\dot{U}_n \text{ is a finite subset of } 2^{<\omega} \text{ such that } \operatorname{diam}([s]) \le \delta \text{ for every } s \in \dot{U}_n \\ \text{and } \sum_{s\in\dot{U}_n} f(\operatorname{diam}([s])) \le \varepsilon \cdot 2^{-n}\text{”}\} \end{aligned}$$

If $P = \mathbb{L}$, we omit the superscript $\mathbb{L}$ to write it as $I^f_{\varepsilon,\delta}$.

Theorem 4.8.10. Let f be a gauge function, $\delta > 0$ and $A \subseteq 2^\omega$. Let $a = \mathcal{H}^f_\delta(A)$. Then $\mathbb{L} \Vdash \mathcal{H}^f_\delta(A) \ge a/2$.

Proof. Suppose $\mathbb{L} \not\Vdash \mathcal{H}^f_\delta(A) \ge a/2$. Then we can take a positive, rational number $\varepsilon < a/2$ and $T \in \mathbb{L}$ and $\langle \dot{U}_n : n \in \omega\rangle \in I^f_{\varepsilon,\delta}$ such that $T \Vdash A \subseteq \bigcup_n \bigcup_{s\in\dot{U}_n}[s]$. By Lemma 4.8.4 (3), we may assume that $T \in \bigcap_n \operatorname{dec}(\dot{U}_n)^*$. Since $A \subseteq \{x \in 2^\omega : T \Vdash x \in \bigcup_n \bigcup_{s\in\dot{U}_n}[s]\}$, we have $\mathcal{H}^f_\delta(A) \le 2\varepsilon < a$ by Lemma 4.8.8, which is a contradiction. □

4.9 Amoeba forcing of Hausdorff measures

In this section, we give a Hausdorff measures version of the amoeba forcing. We expect to be able to separate $\operatorname{add}(\mathcal{N})$ and the additivity of some Hausdorff measure using this forcing notion. But we could not show this.

Let (X, d) be a second countable metric space and f be a continuous gauge function. Fix an open base $\langle N_n : n \in \omega\rangle$ of X.

Definition 4.9.1. For each $i \in [1, \omega)$, we define a forcing poset $P(i)$ whose conditions are all $p \subseteq \omega$ such that $\sum_{n\in p} f(\operatorname{diam}(N_n)) < 1/2^i$. The order of $P(i)$ is defined to be $q \le p$ iff $q \supseteq p$.

Lemma 4.9.2. For every $A \subseteq X$ with $\mathcal{H}^f(A) = 0$, the set of conditions $\{p \in P(i) : A \subseteq \bigcup_{n\in p} N_n\}$ is dense.

Lemma 4.9.3. Define a $P(i)$-name $\dot{A}(i)$ so that $P(i) \Vdash \dot{A}(i) = \bigcup_{p \in \dot{G}} \bigcup_{n \in p} N_n$ holds. Then $P(i)$ forces $\mathcal{H}^f_\infty(\dot{A}(i)) \leq 1/2^i$.

Lemma 4.9.4. $P(i)$ is σ-linked for each $i \in [1, \omega)$.

Proof. For $a \in [\omega]^{<\omega}$, we define a set $P_a(i)$ so that

$$P_a(i) = \left\{ p \in P(i) : a \subseteq p, \frac{1}{2} \sum_{n \in a} f(\mathrm{diam}(N_n)) + \sum_{n \in p \smallsetminus a} f(\mathrm{diam}(N_n)) < \frac{1}{2^{i+1}} \right\}.$$

We claim that each P_a is linked. Let $p, q \in P_a$. Consider $r = p \cup q$. We have

$$\frac{1}{2} \sum_{n \in a} f(\mathrm{diam}(N_n)) + \sum_{n \in p \smallsetminus a} f(\mathrm{diam}(N_n)) < \frac{1}{2^{i+1}},$$
$$\frac{1}{2} \sum_{n \in a} f(\mathrm{diam}(N_n)) + \sum_{n \in q \smallsetminus a} f(\mathrm{diam}(N_n)) < \frac{1}{2^{i+1}}.$$

Adding the sides gives the following inequality:

$$\sum_{n \in a} f(\mathrm{diam}(N_n)) + \sum_{n \in (p \cup q) \smallsetminus a} f(\mathrm{diam}(N_n)) < \frac{1}{2^i},$$

which shows that $p \cup q \in P(i)$.

Finally, we show $P(i) = \bigcup \{P_a(i) : a \in [\omega]^{<\omega}\}$. Let $p \in P(i)$. Then $S := \sum_{n \in p} f(\mathrm{diam}(N_n)) < 1/2^i$. we can take $a \in [\omega]^{<\omega}$ such that $\sum_{n \in p \smallsetminus a} f(\mathrm{diam}(N_n)) < 1/2^i - S$. Put $T = \sum_{n \in p \smallsetminus a} f(\mathrm{diam}(N_n))$, which implies $S + T < 1/2^i$. Then we have

$$\frac{1}{2} \sum_{n \in a} f(\mathrm{diam}(N_n)) + \sum_{n \in p \smallsetminus a} f(\mathrm{diam}(N_n)) = \frac{1}{2}(S - T) + T = \frac{1}{2}(S + T) < \frac{1}{2^{i+1}}.$$

Therefore we showed $p \in P_a(i)$. □

Definition 4.9.5. Let P be the finite support product of $P(i)$ for $i \in [1, \omega)$.

Theorem 4.9.6. P is ccc and P adds a set of reals of f-Hausdorff measure 0 that contains all subsets of X of f-Hausdorff measure 0 coded in V.

Proof. It follows from Lemmas 4.9.2, 4.9.3 and 4.9.4. □

4.10 Open problems

Problem 4.10.1. (1) Is it consistent that $\mathrm{non}(\mathcal{I}_{\mathrm{id}}) < \mathrm{non}(\mathsf{HDZ})$?

(2) Is it consistent that $\mathrm{cov}(\mathsf{HDZ}) < \mathrm{cov}(\mathcal{I}_{\mathrm{id}})$?

Problem 4.10.2. (1) Is it consistent that $\mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathsf{HDZ})$?

(2) Is it consistent that $\mathrm{cof}(\mathsf{HDZ}) < \mathrm{cof}(\mathcal{N})$?

Problem 4.10.3. Do countable support iterations of Laver forcing preserve sets of Hausdorff measure positive?

Problem 4.10.4. Is it consistent that $\mathrm{add}(\mathcal{N}) \neq \mathrm{add}(\mathcal{N}^f)$ for some gauge function f? In particular is this consistency achieved by the forcing notion in Section 4.9?

Chapter 5

Keisler's theorem

The following is an important theorem in model theory proved by Keisler and Shelah. Keisler [Kei64] proved it by assuming GCH, but Shelah [She71] removed that assumption.

Theorem 5.0.1 (Keisler–Shelah)**.** For every (first-order) language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$, the following are equivalent:

(1) $\mathcal{A} \equiv \mathcal{B}$ (that is, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).

(2) There is a nonprincipal ultrafilter $\mathcal{U}$ over an infinite set such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

The following theorem is also known in connection with the above theorem.

Theorem 5.0.2 (Keisler, Golshani and Shelah)**.** The following are equivalent:

(1) The continuum hypothesis.

(2) For every countable language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \mathfrak{c}$, if $\mathcal{A} \equiv \mathcal{B}$ then there is a nonprincipal ultrafilter $\mathcal{U}$ over ω such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

For this theorem, Keisler [Kei64] showed (1) $\Rightarrow$ (2) and Golshani and Shelah [GS23] (2) $\Rightarrow$ (1). In order to analyze these theorems in detail, we introduce the following principle.

Definition 5.0.3. Let κ, μ and λ be infinite cardinals. We define a criterion $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ by

$$\begin{aligned} \mathrm{KT}^{\mu}_{\kappa}(\lambda) \iff & \text{for every language } \mathcal{L} \text{ of size} \leq \mu \text{ and} \\ & \text{all elementarily equivalent } \mathcal{L}\text{-structures } \mathcal{A}, \mathcal{B} \text{ of size} \leq \lambda, \\ & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that } \mathcal{A}^{\mathcal{U}} \simeq \mathcal{B}^{\mathcal{U}}. \end{aligned}$$

We also define a criterion $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ by

$$\begin{aligned} \mathrm{SAT}^{\mu}_{\kappa}(\lambda) \iff & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that} \\ & \text{for every language } \mathcal{L} \text{ of size} \leq \mu \text{ and} \\ & \text{every sequence } \langle \mathcal{A}_i : i < \kappa \rangle \text{ of infinite } \mathcal{L}\text{-structures of size} \leq \lambda, \\ & \text{the ultraproduct } \left(\prod_{i \in \kappa} \mathcal{A}_i \right) / \mathcal{U} \text{ is saturated.} \end{aligned}$$

Keisler–Shelah's theorem means that $\mathrm{KT}^{2^\lambda}_{2^\lambda}(\lambda)$ holds for any infinite cardinal λ.

Keisler's paper also gives an example showing the following.

Fact 5.0.4 (Keisler [Kei64])**.** Let κ be an infinite cardinal. Then $\neg\,\mathrm{KT}^{\kappa^+}_{\kappa}(\kappa^+)$ holds.

We introduce abbreviations for countable languages and ultrafilters on ω.

Definition 5.0.5. Let κ be a cardinal.

(1) We say $\mathrm{KT}(\kappa)$ holds if $\mathrm{KT}^{\aleph_0}_{\aleph_0}(\kappa)$ holds.

(2) We say $\mathrm{SAT}(\kappa)$ holds if $\mathrm{SAT}^{\aleph_0}_{\aleph_0}(\kappa)$ holds.

In this chapter, we prove the implications indicated by thick lines in Figure 5.1.

$$
\begin{tikzcd}
\mathrm{CH} \arrow[rrr] \arrow[dd, leftrightarrow] \arrow[dddr] & & & \mathrm{MA} \arrow[d] \\
 & & & \mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge\ 2^{<\mathfrak{c}} = \mathfrak{c} \arrow[d, leftrightarrow, very thick] \\
\mathrm{SAT}(\mathfrak{c}) \arrow[rr, leftrightarrow, very thick] \arrow[dd, leftrightarrow] & & \mathrm{SAT}(\aleph_1) \arrow[r] \arrow[dd] & \mathrm{SAT}(\aleph_0) \arrow[d, very thick] \\
 & \begin{array}{c}\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \\ \wedge\ \mathrm{cf}(\mathfrak{c}) = \aleph_1\end{array} \arrow[rr] \arrow[dr] & & \mathrm{cov}(\mathcal{M}) = \mathfrak{c} \arrow[d] \\
\mathrm{KT}(\mathfrak{c}) \arrow[rr] & & \mathrm{KT}(\aleph_1) \arrow[r] \arrow[d, very thick] & \mathrm{KT}(\aleph_0) \arrow[d, very thick] \\
 & & \mathfrak{b} = \aleph_1 & \mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}
\end{tikzcd}
$$

Figure 5.1: Implications; thick arrows indicate our results

We use the following fact later.

Fact 5.0.6 ([BS06, Lemma 3.5 and Theorem 3.12])**.** Let $\langle \mathcal{A}_i : i \in \omega \rangle$ be a sequence of structures in a language $\mathcal{L}$ such that each $\mathcal{A}_i$ has size $\leq \mathfrak{c}$. Let $\mathcal{U}$ be an ultrafilter over ω. Then the ultraproduct $\prod_{i\in\omega} \mathcal{A}_i/\mathcal{U}$ has size either finite or $\mathfrak{c}$.

5.1 $\mathrm{SAT}(\aleph_1)$ and $\mathrm{KT}(\aleph_1)$

In this section, we prove that $\mathrm{SAT}(\aleph_1)$ is equivalent to CH and that $\mathrm{KT}(\aleph_1)$ implies $\mathfrak{b} = \aleph_1$.

Theorem 5.1.1. $\mathrm{SAT}(\aleph_1)$ implies CH.

Proof. Assume $\mathrm{SAT}(\aleph_1)$ and $\neg$CH. Take an ultrafilter $\mathcal{U}$ over ω that witnesses $\mathrm{SAT}(\aleph_1)$. Let $\mathcal{A}_* = (\omega_1, <)^\omega/\mathcal{U}$. For $\alpha < \omega_1$, put $\alpha_* = [\langle \alpha, \alpha, \alpha, \dots \rangle]$. Define a set p of formulas with a free variable x by

$$p = \{\ulcorner \alpha_* < x \urcorner : \alpha < \omega_1\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is $\aleph_1 < \mathfrak{c} = |\mathcal{A}_*|$ by $\neg$CH. Thus, by SAT($\aleph_1$), we can take $f\colon \omega \to \omega_1$ such that $[f]$ realizes p. Put $\beta = \sup_{n\in\omega} f(n)$. Now we have $\{n \in \omega : \beta < f(n)\} \in \mathcal{U}$ and this contradicts the definition of β. $\square$

Theorem 5.1.2. $\neg$SAT($\aleph_2$) holds.

Proof. Take an ultrafilter $\mathcal{U}$ over ω that witnesses SAT($\aleph_2$). Let $\mathcal{A}_* = (\omega_2, <)^\omega/\mathcal{U}$. For $\alpha < \omega_1$, put $\alpha_* = [\langle \alpha, \alpha, \alpha, \dots\rangle]$. Define a set p of formulas with a free variable x by

$$p = \{\ulcorner \alpha_* < x < (\omega_1)_* \urcorner : \alpha < \omega_1\}.$$

The remaining argument is the same as Theorem 5.1.1. $\square$

Definition 5.1.3. Let $\mathfrak{mcf} = \min\{\mathrm{cf}(\omega^\omega/\mathcal{U}) : \mathcal{U}$ an ultrafilter over $\omega\}$.

The order of $\omega^\omega/\mathcal{U}$ is the almost domination order modulo $\mathcal{U}$ and $\mathrm{cf}(\omega^\omega/\mathcal{U})$ is the dominating number of this relation. So it is clear that $\mathfrak{b} \leq \mathfrak{mcf} \leq \mathfrak{d}$.

Lemma 5.1.4 ([GS22, Claim 2.2])**.** Let $\mathcal{A}$ be a structure in a language $\mathcal{L} = \{<\}$. Suppose that $a \in \mathcal{A}$ has cofinality ω_1. Let $\mathcal{U}$ be an ultrafilter over ω. Then $a_* = [\langle a, a, a, \dots\rangle]$ has cofinality ω_1 in $\mathcal{A}^\omega/\mathcal{U}$.

Proof. Take an increasing cofinal sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ of points in $\mathcal{A}$ below a. Then $\langle x^*_\alpha : \alpha < \omega_1\rangle$ is an increasing cofinal sequence in $\mathcal{A}_*$, where $x^*_\alpha = [\langle x_\alpha, x_\alpha, x_\alpha, \dots\rangle]$ for each $\alpha < \omega_1$. This can be shown by regularity of ω_1. $\square$

Lemma 5.1.5 ([GS22, Claim 2.4])**.** Let $\mathcal{U}$ be an ultrafilter over ω and $\mathcal{B}_* = (\mathbb{Q}, <)^\omega/\mathcal{U}$. Then for every $a, b \in \mathcal{B}_*$, there is an automorphism on $\mathcal{B}_*$ that sends a to b.

Proof. Consider the map $F\colon \mathbb{Q}^3 \to \mathbb{Q}$ defined by $F(x, y, z) = x - y + z$. Then we have

$$(\forall y, z \in \mathbb{Q})(\text{the map } x \mapsto F(x, y, z) \text{ is an automorphism on } (\mathbb{Q}, <) \text{ that sends } y \text{ to } z).$$

This statement can be written by a first-order formula in the language $\mathcal{L}' = \{<, F\}$. Thus the same statement is true in $(\mathbb{Q}, <, F)^\omega/\mathcal{U}$. The map $F_* : \mathcal{B}_*^3 \to \mathcal{B}_*$ induced by F satisfies that

$$(\forall y, z \in \mathcal{B}_*)(\text{the map } x \mapsto F(x, y, z) \text{ is an automorphism on } (\mathcal{B}_*, <) \text{ that sends } y \text{ to } z).$$ $\square$

Theorem 5.1.6. KT($\aleph_1$) implies $\mathfrak{mcf} = \aleph_1$.

Proof. This proof is based on [GS22, Theorem 2.1]. Assume that $\mathfrak{mcf} \geq \aleph_2$. We shall show $\neg$KT($\aleph_1$).

Let $\mathcal{L} = \{<\}$, $\mathcal{A} = (\mathbb{Q}, <)$ and $\mathcal{B} = (\mathbb{Q} + ((\omega_1 + 1) \times \mathbb{Q}_{\geq 0}), <_\mathcal{B})$. Here $<_\mathcal{B}$ is defined by a lexicographical order and a disjoint union order. $\mathcal{A}$ and $\mathcal{B}$ are dense linear ordered sets, so by completeness of DLO, we have $\mathcal{A} \equiv \mathcal{B}$. Take an ultrafilter $\mathcal{U}$ over ω. Put $\mathcal{A}_* = \mathcal{A}^\omega/\mathcal{U}, \mathcal{B}_* = \mathcal{B}^\omega/\mathcal{U}$.

There is a point a in $\mathcal{B}$ such that $\mathrm{cf}(\mathcal{B}_a) = \aleph_1$, where $\mathcal{B}_a = \{x \in \mathcal{B} : x < a\}$. Then $a_* \in \mathcal{B}_*$ has cofinality $\aleph_1$ by Lemma 5.1.4. Here $a_* = [\langle a, a, a, \dots\rangle]$. On the other hand, we shall show every point in $\mathcal{A}_*$ has cofinality $\geq \mathfrak{mcf}$. If we do this, since we assumed $\mathfrak{mcf} \geq \aleph_2$, we will have $\mathcal{A}_* \not\cong \mathcal{B}_*$.

By Lemma 5.1.5, it suffices to consider the point $0_* = [\langle 0, 0, 0, \dots\rangle]$. Since $\mathbb{Q}$ is symmetrical, we consider $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_\mathcal{U})$.

Now we construct a Tukey morphism $(\varphi, \psi)\colon \mathbf{Cof}(\omega^\omega/\mathcal{U}) \to \mathbf{Cof}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_{\mathcal{U}})$ by

$$\varphi\colon \omega^\omega/\mathcal{U} \to (\mathbb{Q}_{>0})^\omega/\mathcal{U}; [f] \mapsto [\langle 1/(f(n)+1) : n \in \omega\rangle],$$
$$\psi\colon (\mathbb{Q}_{>0})^\omega/\mathcal{U} \to \omega^\omega/\mathcal{U}; [g] \mapsto [\langle \lfloor 1/g(n) - 1\rfloor : n \in \omega\rangle].$$

So we have $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_U) \geq \mathrm{cf}(\omega^\omega/\mathcal{U}, <_{\mathcal{U}})$.

Thus we have $\mathrm{cf}((\mathbb{Q}_{>0})^\omega/\mathcal{U}, >_U) \geq \mathfrak{mcf}$. We are done. □

Corollary 5.1.7. $\mathrm{KT}(\aleph_1)$ implies $\mathfrak{b} = \aleph_1$.

Proof. This follows from Theorem 5.1.6 and the fact that $\mathfrak{b} \leq \mathfrak{mcf}$. □

5.2 $\mathrm{SAT}(\aleph_0)$ and $\mathrm{KT}(\aleph_0)$

In this section, we first briefly mention consistency of $\mathrm{KT}(\aleph_0) + \neg\,\mathrm{KT}(\aleph_1)$. And we prove that $\mathrm{SAT}(\aleph_0)$ is equivalent to $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$.

Fact 5.2.1 ([Bla10, Theorem 7.13])**.** The statement $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is equivalent to MA(countable), that is for every countable poset $\mathbb{P}$ and a family of dense sets $\mathcal{D}$ with $|\mathcal{D}| < \mathfrak{c}$ there is a filter G of $\mathbb{P}$ that intersects all $D \in \mathcal{D}$.

Theorem 5.2.2. $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ implies $\mathrm{KT}(\aleph_0)$.

Proof. [GS22, Theorem 3.3] shows that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) = \aleph_1$ implies $\mathrm{KT}(\aleph_1)$ and the exact same proof works for $\mathrm{KT}(\aleph_0)$ without the assumption $\mathrm{cf}(\mathfrak{c}) = \aleph_1$.

Here we sketch the proof.

Let $\mathcal{L}$ be a countable language and $\mathcal{A}^0$ and $\mathcal{A}^1$ are countable $\mathcal{L}$-structures which are elementarily equivalent.

Enumerate $(\mathcal{A}^i)^\omega$ for $i = 0, 1$ as

$$(\mathcal{A}^i)^\omega = \{f^i_\alpha : \alpha < \mathfrak{c}\}.$$

By a back-and-forth method, we construct a sequence of triples $\langle(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < \mathfrak{c}\rangle$ satisfying:

(1) $g^0_\alpha \in \mathcal{A}^0$,

(2) $g^1_\alpha \in \mathcal{A}^1$,

(3) $\mathcal{U}_\alpha$ is a filter over ω generated by $\aleph_0 + |\alpha|$ sets,

(4) $(\mathcal{U}_\alpha : \alpha < \mathfrak{c})$ is an increasing continuous sequence,

(5) If $\varphi(x_0, \ldots, x_{n-1}$ is an $\mathcal{L}$-formula and $\beta_0, \ldots, \beta_n \leq \alpha$, then the set

$$\{k \in \omega : \mathcal{M}^0 \models \varphi(g^0_{\beta_0}(k), \ldots, g^0_{\beta_{n-1}}(k)) \iff \mathcal{M}^1 \models \varphi(g^1_{\beta_0}(k), \ldots, g^1_{\beta_{n-1}}(k))\}$$

belongs to $\mathcal{U}_{\alpha+1}$.

In the construction, when α is even, we put $g^0_\alpha = f^0_\gamma$ where γ is the least ordinal $f^0_\gamma \notin \{g^0_\beta : \beta < \alpha\}$. And $\mathbb{P}$ is the poset of finite partial functions from ω to $\mathcal{A}^1$. Take a generating set $\mathcal{F}$ of $\mathcal{U}_\alpha$ of size

$\aleph_0 + |\alpha|$. Then by using MA(countable), take a $\mathbb{P}$-generic filter G with respect to a the following family of dense sets of $\mathbb{P}$:

$$D_n = \{p \in \mathbb{P} : n \in \operatorname{dom} p\} \text{ (for } n \in \omega)$$

and

$$\begin{aligned} E_{X, \langle \varphi_\iota : \iota \in I \rangle, \langle \gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota : \iota \in I \rangle} = \{ & p \in \mathbb{P} : (\exists k \in \operatorname{dom}(p) \cap X)(\forall \iota \in I) \\ & (M^0 \models \varphi_\iota(g^0_{\gamma_1^\iota}(k), \dots g^0_{\gamma^\iota_{n_\iota}}(k), g^0_\alpha(k)) \Leftrightarrow \\ & M^1 \models \varphi_\iota(g^1_{\gamma_1^\iota}(k), \dots g^1_{\gamma^\iota_{n_\iota}}(k), p(k))\}), \end{aligned}$$

where $X \in \mathcal{F}$, $\langle \varphi_\iota : \iota \in I \rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma_1^\iota, \dots, \gamma_{n_\iota}^\iota$ for $\iota \in I$ are ordinals less than α. Then putting $g^1_\alpha = \bigcup G$ satisfies the induction hypothesis.

Then the appropriate construction guarantees that $\mathcal{U} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{U}_\alpha$ is an ultrafilter and that the function

$$\langle ([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < \mathfrak{c} \rangle$$

is an isomorphism from $(M^0)^\omega/\mathcal{U}$ to $(M^1)^\omega/\mathcal{U}$. □

Corollary 5.2.3. Assume Con(ZFC). Then Con($\mathsf{ZFC} + \mathrm{KT}(\aleph_0) + \neg\,\mathrm{KT}(\aleph_1)$).

Proof. MA + ¬CH implies $\mathrm{KT}(\aleph_0) \wedge \neg\,\mathrm{KT}(\aleph_1)$ by Theorem 5.1.6 and 5.2.2. □

Fact 5.2.4 ([BJ95, Lemma 2.4.2])**.** $\mathrm{cov}(\mathcal{M}) = \mathfrak{v}^\exists_{\langle \omega : n \in \omega \rangle, \mathrm{id}}$. In other words, $\mathrm{cov}(\mathcal{M}) \geq \kappa$ holds iff $(\forall X \subseteq \omega^\omega$ of size $< \kappa)(\exists S \in \prod_{i \in \omega} [\omega]^{\leq i})(\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$ holds.

Theorem 5.2.5. $\mathrm{SAT}(\aleph_0)$ implies $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

Proof. Take an ultrafilter $\mathcal{U}$ that witnesses $\mathrm{SAT}(\aleph_0)$. Fix $X \subseteq \omega^\omega$ of size $< \mathfrak{c}$. Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$ and for each $i \in \omega$, define a $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\omega]^{\leq i}, \subseteq)$. For each $x \in \omega^\omega$, let $S_x = \langle \{x(i)\} : i \in \omega \rangle$. In the ultraproduct $\mathcal{A}_* = \prod_{i \in \omega} \mathcal{A}_i/\mathcal{U}$, define a set p of formulas of one free variable S by

$$p = \{\ulcorner [S_x] \subseteq S \urcorner : x \in X\}.$$

This p is finitely satisfiable. In order to check this, let $x_0, \dots, x_n$ be finitely many members of X. Define S by $S(m) = \{x_0(m), \dots, x_n(m)\}$ for $m \geq n$. We don't need to care about $S(m)$ for $m < n$. Then this S satisfies $[S_{x_i}] \subseteq [S]$ for all $i \leq n$. Moreover, the number of parameters of p is $< \mathfrak{c}$.

So by $\mathrm{SAT}(\aleph_0)$, we can take $[S] \in \mathcal{A}_*$ that realizes p. Then S fulfills $(\forall x \in X)(\{n \in \omega : x(n) \in S(n)\} \in \mathcal{U})$. Thus $(\forall x \in X)(\exists^\infty n)(x(n) \in S(n))$. □

Theorem 5.2.6. $\mathrm{SAT}(\aleph_0)$ implies $2^{<\mathfrak{c}} = \mathfrak{c}$.

Proof. Take an ultrafilter $\mathcal{U}$ over ω that witnesses $\mathrm{SAT}(\aleph_0)$. Fix $\kappa < \mathfrak{c}$.

Put $\mathcal{L} = \{\subseteq\}$ and define an $\mathcal{L}$-structure $\mathcal{A}$ by $\mathcal{A} = ([\omega]^{<\omega}, \subseteq)$. Put $\mathcal{A}^* = \mathcal{A}^\omega/\mathcal{U}$.

Define a map $\iota \colon \omega^\omega/\mathcal{U} \to \mathcal{A}^*$ by $\iota([x]) = [\langle \{x(n)\} : n \in \omega \rangle]$. By Fact 5.0.6, we have $|\omega^\omega/\mathcal{U}| = \mathfrak{c}$. Take a subset F of $\omega^\omega/\mathcal{U}$ of size κ.

For each $X \subseteq F$, let p_X be a set of formulas with a free variable z defined by

$$p_X = \{\ulcorner \iota(y) \subseteq z \urcorner : y \in X\} \cup \{\ulcorner \iota(y) \not\subseteq z \urcorner : y \in F \setminus X\}$$

Each p_X is finitely satisfiable. In order to check this, take $[x_0], \dots, [x_n] \in X$ and $[y_0], \dots, [y_m] \in F \setminus X$. Put $z(i) = \{x_0(i), \dots, x_n(i)\}$. Then $\iota([x_0]), \dots, \iota([x_n]) \subseteq_U [z]$. In order to prove $\iota([y_j]) \not\subseteq_U [z]$ for each $j \le m$, suppose that $\{i \in \omega : y_j(i) \in z(i)\} \in \mathcal{U}$. Then for each $i \in \omega$, there is a $k_i \le n$ such that $\{i \in \omega : y_j(i) = x_{k_i}(i)\} \in \mathcal{U}$. Then there is a $k \le n$ such that $\{i \in \omega : y_j(i) = x_k(i)\} \in \mathcal{U}$. This implies $[y_j] = [x_k]$, which is a contradiction.

By $\mathrm{SAT}(\aleph_0)$, for each $X \subseteq F$, take $[z_X] \in \mathcal{A}^*$ that realizes p_X. For $X, Y \subseteq F$ with $X \ne Y$, we have $[z_X] \ne [z_Y]$. So $2^\kappa = |\{[z_X] : X \subseteq F\}| \le |\mathcal{A}^*| = \mathfrak{c}$. Therefore we have proved $2^{<\mathfrak{c}} = \mathfrak{c}$. □

Theorem 5.2.7. $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge 2^{<\mathfrak{c}} = \mathfrak{c}$ implies $\mathrm{SAT}(\aleph_0)$.

Proof. This proof is based on [ER72, Theorem 1].

Let $\langle b_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of ω^ω. Let $\langle (\mathcal{L}_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < \mathfrak{c}\rangle$ be an enumeration of triples $(\mathcal{L}, \mathcal{B}, \Delta)$ such that $\mathcal{L}$ is a countable language, $\mathcal{B} = \langle \mathcal{A}_i : i \in \omega\rangle$ is a sequence of $\mathcal{L}$-structures with universe ω and Δ is a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < \mathfrak{c}$. Here $\mathcal{L}^+ = \mathcal{L} \cup \{c_\alpha : \alpha < \mathfrak{c}\}$ where the c_α's are new constant symbols and $\mathrm{Fml}(\mathcal{L}^+)$ is the set of all $\mathcal{L}^+$ formulas with one free variable. Here we used the assumption $2^{<\mathfrak{c}} = \mathfrak{c}$. And ensure each $(\mathcal{L}, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

For $\mathcal{B}_\xi = \langle \mathcal{A}_i^\xi : i \in \omega\rangle$, put $\mathcal{B}_\xi(i) = (\mathcal{A}_i^\xi, b_0(i), b_1(i), \dots)$, which is a $\mathcal{L}^+$-structure.

Let $\langle X_\xi : \xi < \mathfrak{c}\rangle$ be an enumeration of $\mathcal{P}(\omega)$.

We construct a sequence $\langle F_\xi : \xi < \mathfrak{c}\rangle$ of filters inductively so that the following properties hold:

(1) F_0 is the filter consisting of all cofinite subsets of ω.

(2) $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha<\xi} F_\alpha$ for ξ limit.

(3) $X_\xi \in F_{\xi+1}$ or $\omega \setminus X_\xi \in F_{\xi+1}$.

(4) F_ξ is generated by $< \mathfrak{c}$ members.

(5) If

$$\text{for all } \Gamma \subseteq \Delta_\xi \text{ finite}, \{i \in \omega : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi, \tag{$*$}$$

then there is a $f \in \omega^\omega$ such that for all $\varphi \in \Delta_\xi$, $\{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

Suppose we have constructed F_ξ. We construct $F_{\xi+1}$. Let F'_ξ be a generating subset of F_ξ with $|F'_\xi| < \mathfrak{c}$. If $(*)$ is false, let $F_{\xi+1}$ be the filter generated by $F'_\xi \cup \{X_\xi\}$ or $F'_\xi \cup \{\omega \setminus X_\xi\}$. Suppose $(*)$.

Put $\mathbb{P} = \mathrm{Fn}(\omega, \omega) = \{p : p \text{ is a finite partial function from } \omega \text{ to } \omega\}$. For $n \in \omega$, put

$$D_n = \{p \in \mathbb{P} : n \in \mathrm{dom}\, p\}.$$

For $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$, put

$$E_{A,\varphi_1,\dots,\varphi_n} = \{p \in \mathbb{P} : (\exists i \in \mathrm{dom}\, p \cap A)(p(i) \text{ satisfies } \varphi_1, \dots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\}.$$

Each D_n is clearly dense. In order to show that each $E_{A,\varphi_1,\dots,\varphi_n}$ is dense, take $p \in \mathbb{P}$. By $(*)$ and the property $A \in F_\xi$, we can take $i \in A \setminus \mathrm{dom}\, p$ and $k \in \omega$ such that k satisfies $\varphi_1, \dots, \varphi_n$ in $\mathcal{B}_\xi(i)$. Put $q = p \cup \{(i, k)\}$. This is an extension of p in $E_{A,\varphi_1,\dots,\varphi_n}$.

By using MA(countable), take a generic filter $G \subseteq \mathbb{P}$ with respect to above dense sets. Put $f = \bigcup G$. Then $F''_\xi := F'_\xi \cup \{Y_\varphi : \varphi \in \Delta_\xi\}$ satisfies finite intersection property, where $Y_\varphi = \{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\}$. In order to check this, let $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$. Then by genericity, we can take $p \in G \cap E_{A,\varphi_1,\dots,\varphi_n}$. So we can take $i \in \operatorname{dom} p \cap A$ such that $p(i)$ satisfies $\varphi_1, \dots, \varphi_n$ in $\mathcal{B}_\xi(i)$. Then we have $i \in A \cap Y_{\varphi_1} \cap \dots \cap Y_{\varphi_n}$.

Let $F_{\xi+1}$ be the filter generated by $F''_\xi \cup \{X_\xi\}$ or $F''_\xi \cup \{\omega \smallsetminus X_\xi\}$.

We have constructed $\langle F_\xi : \xi < \mathfrak{c} \rangle$. In order to check that the resulting ultrafilter $F = \bigcup_{\xi<\mathfrak{c}} F_\xi$ witnesses $\mathrm{SAT}(\aleph_0)$, let $\mathcal{L}$ and $\mathcal{B} = \langle \mathcal{A}_i : i \in \omega \rangle$ satisfy the assumption of the theorem. Let Δ be a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < \mathfrak{c}$. Assume that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma := \{i \in \omega : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F$. By the regularity of $\mathfrak{c}$, we have $\alpha < \mathfrak{c}$ such that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma \in F_\alpha$. Let $\xi \geq \alpha$ be satisfying $(\mathcal{L}_\xi, \mathcal{B}_\xi, \Delta_\xi) = (\mathcal{L}, \mathcal{B}, \Delta)$. Then by (5), there is a $f \in \omega$ such that for all $\varphi \in \Delta$, $\{i \in \omega : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}(i)\} \in F$. Thus $\prod_{i\in\omega} \mathcal{A}_i / F$ is saturated.

□

5.3 $\mathrm{KT}(\aleph_0)$ implies $\mathfrak{c}^\exists \leq \mathfrak{d}$

In this section, we will show the following theorem. This proof is based on [She92, Theorem 1.1] and [Abr10, Theorem 3.7].

Theorem 5.3.1. $\mathrm{KT}(\aleph_0)$ implies $\mathfrak{c}^\exists \leq \mathfrak{d}$.

Definition 5.3.2. Define a language $\mathcal{L}$ by $\mathcal{L} = \{E, U, V\}$, where E is a binary predicate and U, V are unary predicates. We say a $\mathcal{L}$-structure $M = (|M|, E^M, U^M, V^M)$ is a *bipartite directed graph* if the following conditions hold:

(1) $U^M \cup V^M = |M|$,

(2) $U^M \cap V^M = \varnothing$,

(3) $(\forall x, y \in |M|)(x \; E^M \; y \to (x \in U^M \text{ and } y \in V^M))$.

Definition 5.3.3. For $n, k \in \omega$ with $k \leq n$, define a bipartite directed graph $\Delta_{n,k}$ as follows:

(1) $U^{\Delta_{n,k}} = \{1, 2, 3, \dots, n\}$

(2) $V^{\Delta_{n,k}} = [\{1, 2, 3, \dots, n\}]^{\leq k} \smallsetminus \{\varnothing\}$

(3) For $u \in U^{\Delta_{n,k}}, v \in V^{\Delta_{n,k}}$, $u \; E^{\Delta_{n,k}} \; v$ iff $u \in v$.

Definition 5.3.4. For $n \in \omega$, Let $G_n = \Delta_{n^3,n}$. Let Γ be the disjoint union of $(G_n : n \geq 2)$.

We define a natural order $\lhd$ on Γ by $x \lhd y$ if $m < n$ for $x \in G_m, y \in G_n$. Then Γ is a bipartite directed graph with an order $\lhd$. Put $\mathcal{L}' = \mathcal{L} \cup \{\lhd\}$. From now on, we consider $\mathcal{L}'$-structures which are elementarily equivalent to Γ.

Definition 5.3.5. Let Γ_{NS} be a countable non-standard elementary extension of Γ.

When we say connected components, we mean the connected components when we ignore the orientation of the edges.

Lemma 5.3.6. Let M be an $\mathcal{L}'$-structure that is elementarily equivalent to Γ. Then the connected components of M are precisely the maximal antichains of M with respect to $\lhd$.

Proof. Suppose that $A \subseteq M$ is connected but not an antichain. Then we can find elements $a_0, \dots, a_n \in M$ such that

$$M \models (a_0 E a_1 \vee a_1 E a_0) \wedge \cdots \wedge (a_{n-1} E a_n \vee a_n E a_{n-1}) \wedge$$
$$(a_0 \text{ and } a_n \text{ are comparable with respect to } \triangleleft).$$

By elementarity, we have $n+1$ many elements in Γ that satisfy the same formula. This is a contradiction. So every connected subset in M is an antichain.

Note that any two connected vertexes in Γ have a path of length at most 4. Thus we have

$$\Gamma \models (\forall a, b)((a \text{ and } b \text{ are incomparable with respect to } \triangleleft)$$
$$\rightarrow (\text{there is a path between } a \text{ and } b \text{ with length at most 4})).$$

By elementarity, the same formula holds in M. So every antichain in M is connected.

Therefore the connected components of M are precisely the maximal antichains of M with respect to $\triangleleft$. $\square$

Therefore, $\triangleleft$ induces an order on the connected components of M and it is denoted also by $\triangleleft$.

Lemma 5.3.7. Every infinite connected component C of Γ_{NS} satisfies the following:

$$(\forall F \subseteq C \cap U \text{ finite})(\exists v \in C \cap V)(v \text{ has an edge to each point in } F).$$

Proof. Let $F = \{u_1, \dots, u_n\}$. Observe that

$$\begin{aligned}\Gamma \models (\forall x_1) \dots (\forall x_n)[&x_1, \dots, x_n \text{ are points in } U \text{ and belong to} \\ &\text{the same connected component and} \\ &\text{the index of this connected component is } \geq n \\ &\rightarrow (\exists y)[y \text{ belongs to this component, } y \in V \text{ and } x_1, \dots, x_n \; E \; y]].\end{aligned}$$

By elementarity, Γ_{NS} satisfies the same formula. $\square$

Lemma 5.3.8. Let $\langle \Delta_n : n \in \omega \rangle$ be a sequence of bipartite directed graphs with $|U^{\Delta_n}| = |V^{\Delta_n}| = \aleph_0$. Suppose that for each $n \in \omega$,

$$(\forall F \subseteq U^{\Delta_n} \text{ finite})(\exists v \in V^{\Delta_n})(v \text{ has an edge to each point in } F).$$

Then for every ultraproduct $R := \prod_{n \in \omega} \Delta_n / \mathcal{V}$, we have

$$(\exists \langle v_i : i < \mathfrak{d} \rangle \text{ with each } v_i \in V^R)(\forall u \in U^R)(\exists i < \mathfrak{d})(u \; E^R \; v_i).$$

Proof. We may assume that each $U^{\Delta_n} = \omega$. Let $\{f_i : i < \mathfrak{d}\}$ be a cofinal subset of $(\omega^\omega, <^*)$. For each $n, m \in \omega$, take $v_{n,m} \in V^{\Delta_n}$ that is connected with first m points in U^{Δ_n}. For $i < \mathfrak{d}$, put

$$v_i = [\langle v_{n, f_i(n)} : n \in \omega \rangle].$$

Let $[u] \in U^R$. Consider u as an element of ω^ω. Take f_i that dominates u. Then we have

$$\{n \in \omega : u(n)\ E^{\Delta_n}\ v_{n,f_i(n)}\} \in \mathcal{V}.$$

Therefore $[u]\ E^R\ v_i$. $\square$

Lemma 5.3.9. Let $\mathcal{V}$ be an ultrafilter over ω and put $Q = (\Gamma_{\mathrm{NS}})^\omega/\mathcal{V}$. Then there exist cofinally many connected components C with respect to $\triangleleft$ such that

$$(\exists \langle v_i : i < \mathfrak{d}\rangle \text{ with each } v_i \in C \cap V^Q)(\forall u \in C \cap U^Q)(\exists i < \mathfrak{d})(u\ E^Q\ v_i).$$

Proof. Fix a connected component C_0 of Q and $[x_0] \in C_0$. Then for each $n \in \omega$, there is an infinite component C_n above $x_0(n)$. Now

$$C = \{[x] \in Q : x \in (\Gamma_{\mathrm{NS}})^\omega \text{ and } (\forall n \in \omega)(x(n) \in C_n)\}.$$

is a connected component of Q above C_0. Since C can be viewed as $C = \prod_{n\in\omega} C_n/\mathcal{V}$, the conclusion of the lemma holds for C by Lemma 5.3.7 and Lemma 5.3.8. $\square$

Lemma 5.3.10. Let $\kappa < \mathfrak{c}^\exists$ and $\mathcal{U}$ be an ultrafilter over ω and put $P = \Gamma^\omega/\mathcal{U}$. Then for every C in a final segment of connected components of P, we have

$$(\forall \langle v_i : i < \kappa\rangle \text{ with each } v_i \in C \cap V^P)(\exists u \in C \cap U^P)(\forall i < \kappa)(u\ \not\!E^P\ v_i).$$

Proof. Let $f\colon \omega \to \Gamma$ satisfy $f(n) \in G_n$ for all n. Let C_0 be the connected component that $[f]$ belongs to. Take a connected component C such that $C_0 \triangleleft C$ and an element $[g] \in C$. Take a function $h\colon \omega \to \omega$ such that $\{n \in \omega : g(n) \in G_{h(n)}\} \in \mathcal{U}$. Then $A := \{n \in \omega : h(n) \geq n\} \in \mathcal{U}$. Put $h'(n) = \max\{h(n), n\}$.

Take $\langle [v_i] : i < \kappa\rangle$ with each $[v_i] \in C \cap V^P$. Then we have

$$B_i := \{n \in \omega : v_i(n) \in G_{h(n)} \cap V^\Gamma\} \in \mathcal{U}.$$

Take v_i' such that $v_i'(n) = v_i(n)$ for $n \in A \cap B_i$ and $v_i'(n) \in [h'(n)^3]^{\leq h'(n)}$ for $n \in \omega$. The assumption $\kappa < \mathfrak{c}^\exists$ and the calculation

$$\sum_{n\geq 1} \frac{h'(n)}{h'(n)^3} = \sum_{n\geq 1} \frac{1}{h'(n)^2} \leq \sum_{n\geq 1} \frac{1}{n^2} < \infty$$

give a $x \in \prod h'$ such that for all $i < \kappa$, $(\forall^\infty n)(x(n) \notin v_i'(n))$. For each $i < \kappa$, take n_i such that $(\forall n \geq n_i)(x(n) \notin v_i'(n))$.

Take a point $[u] \in C \cap U^P$ such that $u(n) = x(n)$ for all $n \in A$. Then for all $i < \kappa$ we have

$$\{n \in \omega : u(n)\ \not\!E^\Gamma\ v_i(n)\} \supseteq A \cap B_i \cap [n_i, \omega) \in \mathcal{U}.$$

Therefore $[u]\ \not\!E^P\ [v_i]$ for all $i < \kappa$. $\square$

Assume that $\mathfrak{d} < \mathfrak{c}^\exists$. Then by Lemma 5.3.10 and Lemma 5.3.9, for any two ultrafilters $\mathcal{U}, \mathcal{V}$ over ω, we have $\Gamma^\omega/\mathcal{U} \not\simeq (\Gamma_{\mathrm{NS}})^\omega/\mathcal{V}$. So $\neg\,\mathrm{KT}(\aleph_0)$ holds. We have proved Theorem 5.3.1.

Fact 5.3.11 ([KM22, Lemma 2.3])**.** $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{c}^\exists$.

Corollary 5.3.12. In the random model, $\neg\,\mathrm{KT}(\aleph_0)$ holds. $\square$

Proof. This corollary holds since $\aleph_1 = \mathfrak{d} < \mathrm{cov}(\mathcal{N}) = \mathfrak{c}$ in the random model. $\square$

Remark 5.3.13. $\mathfrak{v}^\forall \leq \mathfrak{c}^\exists$ follows from [KM22, Lemma 2.6]. So the implication $\mathrm{KT}(\aleph_0) \implies \mathfrak{d} \geq \mathfrak{c}^\exists$ strengthens the implication $\mathrm{KT}(\aleph_0) \implies \mathfrak{d} \geq \mathfrak{v}^\forall$.

Remark 5.3.14. In [She92], Shelah constructed a creature forcing that forces the following statements:

(1) There are a finite language $\mathcal{L}$ and countable $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \equiv \mathcal{B}$ such that for all ultrafilters $\mathcal{U}, \mathcal{V}$ over ω, we have $\mathcal{A}^\omega/\mathcal{U} \not\simeq \mathcal{B}^\omega/\mathcal{V}$.

(2) There is an ultrafilter $\mathcal{U}$ over ω such that for every countable language $\mathcal{L}$ and any sequence $\langle(\mathcal{A}_n, \mathcal{B}_n) : n \in \omega\rangle$ of pairs of finite $\mathcal{L}$-structures, if $\prod_{n\in\omega} \mathcal{A}_n/\mathcal{U} \equiv \prod_{n\in\omega} \mathcal{B}_n/\mathcal{U}$, then these ultraproducts are isomorphic.

Shelah himself pointed out in [She92, Remark 2.2] item 2 holds in the random model. On the other hand, we have proved item 1 also holds in the random model. Therefore both of above two statements hold in the random model.

5.4 $\mathrm{KT}(\aleph_1)$ in forcing extensions

A theorem by Golshani and Shelah [GS22] states that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} \wedge \mathrm{cf}(\mathfrak{c}) = \aleph_1$ implies $\mathrm{KT}(\aleph_1)$. In [GS22], it was also proved that $\mathrm{cf}(\mathfrak{c}) = \aleph_1$ is not necessary for $\mathrm{KT}(\aleph_1)$. In this section, we prove that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is also not necessary for $\mathrm{KT}(\aleph_1)$.

Theorem 5.4.1. Let $\lambda > \aleph_1$ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1\rangle$ be a finite support forcing iteration. Suppose that for all $\alpha < \omega_1$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is ccc and $|\dot{\mathbb{Q}}_\alpha| \leq \lambda$". And suppose that for all even $\alpha < \omega_1$, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}_\lambda$. Here $\mathbb{C}_\lambda$ denotes the Cohen forcing adjoining λ many Cohen reals. Then, $\Vdash_{\omega_1} \mathrm{KT}(\aleph_1)$.

Proof. This proof is based on [GS22, Theorem 3.3].

Let G be a $(V, \mathbb{P}_{\omega_1})$-generic filter.

Let $\mathcal{L}$ be a countable language and $M^0 \equiv M^1$ be two $\mathcal{L}$-structures of size $\leq \aleph_1$ in $V[G]$. Take sequences $\langle M^l_i : i < \omega_1\rangle$ for $l = 0, 1$ that are increasing and continuous such that each M^l_i is countable elementary substructure of M^l and $M^l = \bigcup_{i<\omega_1} M^l_i$. We can take an increasing sequence $\langle \alpha_i : i < \omega_1\rangle$ of even ordinals such that $M^l_i \in V[G_{\alpha_i+1}]$ for every $l < 2$ and $i < \omega_1$.

For $i < \omega_1$ and $\beta < \lambda$, let c^i_β be the β-th Cohen real added by $\dot{\mathbb{Q}}_{\alpha_i}$.

Take an enumeration $\langle X_\gamma : \gamma < \lambda \cdot \omega_1\rangle$ of $\mathcal{P}(\omega)$ such that $\langle X_\gamma : \gamma < \lambda \cdot (i+1)\rangle \in V[G_{\alpha_i+1}]$ for every $i < \omega_1$. We can take such a sequence. The reason for this is that we can take $\langle \dot{X}_\gamma : \lambda \cdot i \leq \gamma < \lambda \cdot (i+1)\rangle$ as an enumeration of $\mathbb{P}_{\alpha_i+1}$ nice names for subsets of ω and put $X_\gamma = (\dot{X}_\gamma)^G$.

For each $l < 2$, take an enumeration $\langle f^l_\gamma : \gamma < \lambda \cdot \omega_1\rangle$ of $(M^l)^\omega$ such that $f^l_{\lambda\cdot i+\beta} \in (M^l_i)^\omega$ for every $i < \omega_1$ and $\beta < \lambda$ and $\langle f^l_\gamma : \gamma < \lambda \cdot (i+1)\rangle \in V[G_{\alpha_i+1}]$.

For $\lambda' < \lambda$, let $G_{\alpha_i,\lambda'}$ denote $G \cap (\mathbb{P}_{\alpha_i} * \mathbb{C}_{\lambda'})$.

Now we construct a sequence of quadruples $\langle(\mathcal{U}_\gamma, g^0_\gamma, g^1_\gamma, \lambda_\gamma) : \gamma < \lambda \cdot \omega_1\rangle$ by induction so that the following properties hold.

(1) Each $\mathcal{U}_\gamma$ is a filter over ω.

(2) For every $l < 2$, $i < \omega_1$, $\beta < \lambda$ and $\gamma = \lambda \cdot i + \beta$, $g^l_\gamma \in (M^l_i)^\omega \cap V[G_{\alpha_i,\lambda_\gamma}]$.

(3) For every $l < 2$ and $i < \omega_1$, $\langle g^l_\gamma : \gamma < \lambda \cdot (i+1)\rangle \in V[G_{\alpha_i+1}]$.

(4) Each λ_γ is an ordinal below λ. For $\lambda \cdot i \leq \gamma \leq \gamma' < \lambda \cdot (i+1)$, we have $\lambda_\gamma \leq \lambda_{\gamma'}$.

(5) For $i < \omega_1$ and $l < 2$, $\{g^l_\gamma : \gamma < \lambda \cdot i\} = \{f^l_\gamma : \gamma < \lambda \cdot i\}$.

(6) If $\lambda \cdot i \leq \gamma < \lambda \cdot (i+1)$, then $\mathcal{U}_\gamma \in V[G_{\alpha_i}, \lambda_\gamma]$.

(7) If $\gamma < \delta < \lambda \cdot \omega_1$, then $\mathcal{U}_\gamma \subseteq \mathcal{U}_\delta$.

(8) If $\gamma < \lambda \cdot \omega_1$ is a limit ordinal, then $\mathcal{U}_\gamma = \bigcup_{\delta<\gamma} \mathcal{U}_\delta$.

(9) $X_\gamma \in \mathcal{U}_{\gamma+1}$ or $\omega \smallsetminus X_\gamma \in \mathcal{U}_{\gamma+1}$.

(10) If $\varphi(x_1, \ldots, x_n)$ is a $\mathcal{L}$-formula, $\gamma = \lambda \cdot i + \beta$ and $\gamma_1, \ldots, \gamma_n \leq \gamma$, then $Y_{\varphi,\gamma_1,\ldots,\gamma_n}$ defined below belongs to $\mathcal{U}_{\gamma+1}$:

$$Y_{\varphi,\gamma_1,\ldots,\gamma_n} = \{k \in \omega : M^0_i \models \varphi(g^0_{\gamma_1}(k), \ldots, g^0_{\gamma_n}(k)) \Leftrightarrow M^1_i \models \varphi(g^1_{\gamma_1}(k), \ldots, g^1_{\gamma_n}(k))\}$$

(*Construction*) First we let U_0 be the set of cofinite subsets of ω.

Suppose that $\langle \mathcal{U}_\delta : \delta \leq \gamma\rangle$ and $\langle g^0_\delta, g^1_\delta, \lambda_\delta : \delta < \gamma\rangle$ are defined. Now we will define $g^0_\gamma, g^1_\gamma, \lambda_\gamma$ and $\mathcal{U}_{\gamma+1}$. Take i and β such that $\gamma = \lambda \cdot i + \beta$.

Suppose that γ is even.

Let $g^0_\gamma = f^0_{\varepsilon_\gamma}$, where ε_γ is the minimum ordinal such that $f^0_{\varepsilon_\gamma}$ does not belong to $\{g^0_\delta : \delta < \gamma\}$.

Take $\lambda' < \lambda$ such that $M^0_i, M^1_i, \langle g^0_\delta : \delta \leq \gamma\rangle, \langle g^1_\delta : \delta < \gamma\rangle \in V[G_{\alpha_i,\lambda'}]$. Put $\lambda_\gamma = \lambda' + 1$. Take a bijection $\pi^1_i : \omega \to M^1_i$ in $V[G_{\alpha_i,\lambda'}]$. Define g^1_γ by $g^1_\gamma = \pi^1_i \circ c^i_{\lambda'}$.

Put $\mathcal{Y} = \{Y_{\varphi,\gamma_1,\ldots,\gamma_n} : \varphi(x_1, \ldots, x_n)$ is a $\mathcal{L}$-formula and $\gamma_1, \ldots, \gamma_n \leq \gamma\}$. Now we show $\mathcal{U}_\gamma \cup \mathcal{Y}$ has the finite intersection property. In order to show it, let $X \in \mathcal{U}_\gamma$, $\langle \varphi_\iota : \iota \in I\rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma^\iota_1, \ldots, \gamma^\iota_{n_\iota}$ for $\iota \in I$ are ordinals that are less than γ. It suffices to show that the set $D \in V[G_{\alpha_i,\lambda'}]$ defined below is a dense subset of $\mathbb{C}$:

$$\begin{aligned} D = \{p \in \mathbb{C} : (\exists k \in \mathrm{dom}(p) \cap X)(\forall \iota \in I) \\ M^0_i \models \varphi_\iota(g^0_{\gamma^\iota_1}(k), \ldots g^0_{\gamma^\iota_{n_\iota}}(k), g^0_\gamma(k)) \Leftrightarrow M^1_i \models \varphi_\iota(g^1_{\gamma^\iota_1}(k), \ldots g^1_{\gamma^\iota_{n_\iota}}(k), \pi^1_i(p(k)))\}. \end{aligned}$$

We now prove this. Let $p \in \mathbb{C}$.

For each $k \in \omega$ and $\iota \in I$, put

$$v(k, \iota) = \begin{cases} 1 & \text{if } M^0_i \models \varphi_\iota(g^0_{\gamma^\iota_1}(k), \ldots, g^0_{\gamma^\iota_{n_\iota}}(k), g^0_\gamma(k)) \\ 0 & \text{otherwise.} \end{cases}$$

And for each $k \in \omega$ put

$$v(k) = \langle v(k, \iota) : \iota \in I\rangle.$$

Then by finiteness of ${}^I 2$, for some $v_0 \in {}^I 2$, we have $\omega \smallsetminus v^{-1}(v_0) \notin \mathcal{U}_\gamma$.

For each $\iota \in I$, put

$$\varphi^+_\iota(x^\iota_1, \ldots, x^\iota_{n_\iota}, y) \equiv \begin{cases} \varphi_\iota(x^\iota_1, \ldots, x^\iota_{n_\iota}, y) & \text{if } v_0(\iota) = 1 \\ \neg\varphi_\iota(x^\iota_1, \ldots, x^\iota_{n_\iota}, y) & \text{otherwise.} \end{cases}$$

Put

$$\psi \equiv \exists y \bigwedge_{\iota\in I} \varphi_\iota^+(x_1^\iota, \ldots, x_{n_\iota}^\iota, y).$$

Then by the induction hypothesis (5), $Y_{\psi,\langle\gamma_1^\iota,\ldots\gamma_{n_\iota}^\iota:\iota\in I\rangle} \in \mathcal{U}_\gamma$. So take $k \in X \cap v^{-1}(v_0) \cap Y_{\psi,\langle\gamma_1^\iota,\ldots\gamma_{n_\iota}^\iota:\iota\in I\rangle} \setminus \mathrm{dom}(p)$.

Since $M_i^0 \models \psi(\langle g^0_{\gamma_1^\iota}(k), \ldots g^0_{\gamma^\iota_{n_\iota}}(k) : \iota \in I\rangle)$, we have $M_i^1 \models \psi(\langle g^1_{\gamma_1^\iota}(k), \ldots g^1_{\gamma^\iota_{n_\iota}}(k) : \iota \in I\rangle)$.

By the definition of ψ, we can take $y \in M_i^1$ such that $M_i^1 \models \varphi_\iota^+(g^1_{\gamma_1^\iota}(k), \ldots, g^1_{\gamma^\iota_{n_\iota}}(k), y)$ for every $\iota \in I$. We now put $q = p \cup \{(k, (\pi_i^1)^{-1}(y))\} \in \mathbb{C}$. This witnesses deneseness of D.

Now we define $\mathcal{U}_{\gamma+1}$ as the filter generated by $\mathcal{U}_\gamma \cup \mathcal{Y} \cup \{X_\gamma\}$ or the filter generated by $\mathcal{U}_\gamma \cup \mathcal{Y} \cup \{\omega \setminus X_\gamma\}$.

When γ is odd, do the same construction above except for swapping 0 and 1. Since the above construction below $\lambda \cdot (i+1)$ can be performed in $V[G_{\alpha_i+1}]$, (3) in the induction hypothesis holds. (*End of Construction.*)

Now we put $\mathcal{U} = \bigcup_{\gamma<\lambda\cdot\omega_1} \mathcal{U}_\gamma$, which is an ultrafilter over ω. Then the function

$$\langle ([g^0_\gamma]_U, [g^1_\gamma]_U) : \gamma < \lambda \cdot \omega_1 \rangle$$

witnesses $(M^0)^\omega/\mathcal{U} \simeq (M^1)^\omega/\mathcal{U}$. $\square$

Corollary 5.4.2. $\mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \mathrm{cof}(\mathcal{N}) = \aleph_1 < \mathfrak{c} + \mathrm{KT}(\aleph_1))$.

Proof. Let $\mathbb{A}$ denote the amoeba forcing. Let $\lambda > \aleph_1$ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1 \rangle$ be a finite support forcing iteration such that for all even $\alpha < \omega_1$ we have $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}_\lambda$ and for all odd $\alpha < \omega_1$ we have $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{A}$.

Then $\mathbb{P}_{\omega_1} \Vdash \mathrm{KT}(\aleph_1)$ by Theorem 5.4.1.

Moreover, we have $\mathrm{cof}(\mathcal{N}) = \aleph_1$ since the amoeba forcing $\mathbb{A}$ adds a null set containing all null sets coded in the ground model (see [BJ95, p. 106]). $\square$

5.5 Uncountable cases

In this section, we discuss the principles introduced in the previous sections. The case where the cardinality μ of the language and the cardinality κ of the underlying set of the ultrafilter are both $\aleph_0$ was analyzed in detail. Here, the more general case is investigated. However, most of the results are naive generalisations of the arguments in the previous sections.

Lemma 5.5.1. Let $\kappa \leq \kappa'$ be two infinite cardinals. Then $\mathrm{KT}^\mu_\kappa(\lambda)$ implies $\mathrm{KT}^\mu_{\kappa'}(\lambda)$.

Proof. Fix a language $\mathcal{L}$ of size $\leq \mu$ and two elementarily equivalent $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ of size $\leq \lambda$. By $\mathrm{KT}^\mu_\kappa(\lambda)$, we can take a uniform ultrafilter $\mathcal{U}$ on κ. Fix a uniform ultrafilter $\mathcal{V}$ on κ'. Then the ultrapowers of $\mathcal{A}$ and $\mathcal{B}$ by the ultrafilter $\mathcal{U} * \mathcal{V}$ are isomorphic. $\square$

Lemma 5.5.2. (1) $\mathrm{KT}^\mu_\kappa(\lambda)$ implies there exists a regular ultrafilter witnessing $\mathrm{KT}^\mu_\kappa(\lambda)$.

(2) If $\lambda \geq \kappa$, then every witness for $\mathrm{SAT}^\mu_\kappa(\lambda)$ is a regular ultrafilter.

Proof. First, we show (1). Take an ultrafilter $\mathcal{U}$ on κ witnessing $\mathrm{KT}^\mu_\kappa(\lambda)$. Take a regular ultrafilter $\mathcal{V}$ on κ. Then the product ultrafilter $\mathcal{U} * \mathcal{V}$ is regular and witnesses $\mathrm{KT}^\mu_\kappa(\lambda)$.

Next we show (2). Take a witness $\mathcal{U}$ for $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$. Let $M = ([\kappa]^{<\aleph_0}, \subseteq)$ and consider $M_* = M^{\kappa}/\mathcal{U}$. By an easy diagonal argument, we have $|M_*| \geq \kappa^+$. Define a set of formulas p with a free variable x by

$$p = \{\ulcorner \{\alpha\}_* \subseteq x \urcorner : \alpha < \kappa\},$$

where $\{\alpha\}_*$ is the equivalence class of the constant sequence of $\{\alpha\}$. It can be easily checked that p is finitely satisfiable and the number of parameters of p is κ, which is smaller than $|M_*|$. Therefore, by $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$, we can take $f \colon \kappa \to M$ such that $[f]$ satisfies p. This f clearly satisfies $\{i \in \kappa : \alpha \in f(i)\} \in \mathcal{U}$ for every $\alpha < \kappa$. Thus, $\mathcal{U}$ is a regular ultrafilter. □

Lemma 5.5.3. $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ implies $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ for every $\lambda \leq 2^{\kappa}$.

Proof. By regularity (Lemma 5.5.2), the ultrapowers have same cardinality. Thus uniqueness of saturated models implies this lemma. □

Lemma 5.5.4. $\neg\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^{++})$.

Proof. Take a witness $\mathcal{U}$ of $\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^{++})$. Let $\mathcal{A} = (\kappa^{++}, <)$ and $\mathcal{A}_* = \mathcal{A}^{\mathcal{U}}$. We have $|\mathcal{A}_*| \geq |\mathcal{A}| = \kappa^{++}$. Consider the following set p of formulas with one free variable x:

$$p = \{\ulcorner \alpha_* < x < (\kappa^+)_* \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is κ^+. Thus, by $\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^{++})$, we can take $f \colon \kappa \to \kappa^+$ such that $[f]$ realizes p. Put $\beta = \sup_{\alpha<\kappa} f(\alpha)$. By $\beta_* < [f]$, we have $\{\alpha < \kappa : \beta < f(\alpha)\} \in \mathcal{U}$. This contradicts the choice of β. □

Lemma 5.5.5. $\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^+)$ implies $2^{\kappa} = \kappa^+$.

Proof. Take a witness $\mathcal{U}$ of $\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^+)$ and assume $\kappa^+ < 2^{\kappa}$. Let $\mathcal{A}_* = (\kappa^+, <)^{\mathcal{U}}$. We have $|\mathcal{A}_*| = 2^{\kappa}$ since $\mathcal{U}$ is regular (Lemma 5.5.2). Consider the following set p of formulas with one free variable x:

$$p = \{\ulcorner \alpha_* < x \urcorner : \alpha < \kappa^+\}.$$

This p is finitely satisfiable and the number of parameters occurring in p is equal to κ^+, which is smaller than 2^{κ}. Thus, by $\mathrm{SAT}^{\aleph_0}_{\kappa}(\kappa^+)$, we can take $f \colon \kappa \to \kappa^+$ such that $[f]$ realizes p. Then, this f is unbounded, which contradicts that κ^+ is regular. □

Lemma 5.5.6. $\neg\,\mathrm{KT}^{\aleph_0}_{\kappa}(\kappa^{++})$.

Proof. This proof is based on [Tsu22]. Let $(\mathcal{M}, <)$ be a linearly ordered set with cofinality κ^{++}. We define an increasing continuous sequence $\langle A_i : i \leq \kappa^{++}\rangle$ of subsets of $\mathcal{M}$ such that:

(1) For every $i \leq \kappa^{++}$, A_i is an elementary substructure of $\mathcal{M}$.

(2) For every $i < \kappa^{++}$, there is $a_i \in A_{i+1}$ such that for every $b \in A_i$, we have $b < a_i$.

(3) For every $i \leq \kappa^{++}$, we have $|A_i| \leq |i| + \aleph_0$.

We show that the pair of A_{κ^+} and $A_{\kappa^{++}}$ is a counterexample of $\mathrm{KT}_{\kappa}(\kappa^{++})$. Let $\mathcal{U}$ be an ultrafilter on κ.

We claim that $(A_{\kappa^+})^{\mathcal{U}}$ has a cofinal increasing sequence of length κ^+. In fact, $\langle (a_i)_* : i < \kappa^+ \rangle$ is a cofinal increasing sequence. In order to show it, take $[f] \in (A_{\kappa^+})^{\mathcal{U}}$. For each $\alpha < \kappa$, we can take $i_\alpha < \kappa^+$ such that $f(\alpha) \in A_{i_\alpha}$. Then $i = \sup_{\alpha<\kappa} i_\alpha$ satisfies $[f] < a_i$.

On the other hand, in $(A_{\kappa^{++}})^{\mathcal{U}}$, every κ^+-sequence is bounded. In order to check it, take $\langle b_i : i < \kappa^+ \rangle$. We write b_i as $b_i = [f_i]$, where $f_i \colon \kappa \to A_{\kappa^{++}}$. Since the set $\{f_i(\alpha) : i < \kappa^+, \alpha < \kappa\}$ has size less than or equal to κ^+, we can take $\beta < \kappa^{++}$ such that all the elements of this set belong to A_β. Then a_β is a bound of all b_i.

So we have $(A_{\kappa^+})^{\mathcal{U}} \not\cong (A_{\kappa^{++}})^{\mathcal{U}}$. $\square$

Theorem 5.5.7. Let κ and μ be infinite cardinals satisfying $\mu \le \kappa$. Then the following are equivalent.

(1) $2^\kappa = \kappa^+$.

(2) $\mathrm{SAT}^\mu_\kappa(2^\kappa)$.

(3) $\mathrm{SAT}^\mu_\kappa(\kappa^+)$

(4) $\mathrm{KT}^\mu_\kappa(2^\kappa)$.

Proof. Recall that there is a κ^+-good ultrafilter U on κ. That is, for every language $\mathcal{L}$ of size $\le \kappa$, all U-ultraproducts of $\mathcal{L}$-structures are κ^+-saturated. The implication $2^\kappa = \kappa^+ \implies \mathrm{SAT}^\mu_\kappa(2^\kappa)$ follows from this fact.

The implication $\mathrm{SAT}^\mu_\kappa(\kappa^+) \implies 2^\kappa = \kappa^+$ is just Lemma 5.5.5.

The implication $\mathrm{KT}^\mu_\kappa(2^\kappa) \implies 2^\kappa = \kappa^+$ follows from Lemma 5.5.6. $\square$

Theorem 5.5.8. Let κ be a regular cardinal. Then $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$ implies $\mathfrak{b}_\kappa = \kappa^+$.

Proof. Take the same structure $\mathcal{M}$ as in Lemma 5.5.6. Consider two elementary substructures A_κ and A_{κ^+}.

Take a regular ultrafilter $\mathcal{U}$ on κ that witnesses $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$. As we saw in Lemma 5.5.6, we have $\mathrm{cf}((A_{\kappa^+})^{\mathcal{U}}) = \kappa^+$.

On the other hand, we have $\mathrm{cf}(A_\kappa) = \kappa$. So it holds that $\mathrm{cf}((A_\kappa)^{\mathcal{U}}) = \mathrm{cf}(\kappa^\kappa/\mathcal{U})$.

Since the ultrafilter $\mathcal{U}$ is uniform, we have $\mathfrak{b}_\kappa \le \mathrm{cf}(\kappa^\kappa/\mathcal{U})$.

By $\mathrm{KT}^{\aleph_0}_\kappa(\kappa^+)$, the two models $(A_\kappa)^{\mathcal{U}}$ and $(A_{\kappa^+})^{\mathcal{U}}$ are isomorphic. So we have $\mathfrak{b}_\kappa \le \mathrm{cf}(\kappa^\kappa/\mathcal{U}) = \kappa^+$. The other inequality is obvious. $\square$

Theorem 5.5.9. $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$ implies $2^{<2^\kappa} = 2^\kappa$.

Proof. Fix a witness $\mathcal{U}$ for $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$. Let $\lambda < 2^\kappa$. Define a language $\mathcal{L}$ and $\mathcal{L}$-structure $\mathcal{A}$ by $\mathcal{L} = \{\subseteq\}$ and $\mathcal{A} = ([\kappa]^{<\omega}, \subseteq)$. We have $|\mathcal{A}| = \kappa$. Put $\mathcal{A}_* = \mathcal{A}^\kappa/\mathcal{U}$. Since $\mathcal{U}$ is regular (Lemma 5.5.2), we have $|\mathcal{A}_*| = \kappa^\kappa = 2^\kappa$. Let $\iota \colon \kappa^\kappa/\mathcal{U} \to \mathcal{A}_*$ be the function defined by:

$$\iota([x]) = [\langle \{x(\alpha)\} : \alpha < \kappa \rangle].$$

Fix $F \subseteq \kappa^\kappa/\mathcal{U}$ of size λ. For $X \subseteq F$, we define a set $p_X(z)$ of formulas with a free variable z by:

$$p_X(z) = \{\ulcorner \iota(y) \subseteq z \urcorner : y \in X\} \cup \{\ulcorner \iota(y) \not\subseteq z \urcorner : y \in F \setminus X\}.$$

Each $p_X(z)$ is finitely satisfiable and the number of parameters occurring in $p_X(z)$ is λ. Therefore, by $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$, for each $X \subseteq F$, we can take $[z_X] \in \mathcal{A}_*$ satisfying $p_X(z)$. For distinct $X, Y \subseteq F$, we have

$[z_X] \neq [z_Y]$. Thus we have $2^\lambda = |\{[z_X] : X \subseteq F\}| \leq \mathcal{A}_* = 2^\kappa$. Since $\lambda < 2^\kappa$ was arbitrary chosen, we have $2^{<2^\kappa} = 2^\kappa$. $\square$

Theorem 5.5.10. Let κ be a regular cardinal. Let μ be a cardinal less than 2^κ. Then $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$ implies $\mathrm{KT}^\mu_\kappa(\kappa)$.

Proof. Note that the assumption $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$ is equivalent to $\mathrm{MA}_{<2^\kappa}(\mathrm{Fn}_\kappa(\kappa, 2))$.

Fix a enumeration of 2^κ.

Let $\mathcal{L}$ be a language of size $\leq \mu$ and $\mathcal{A}^0$ and $\mathcal{A}^1$ are $\mathcal{L}$-structures of size $\leq \kappa$ which are elementarily equivalent.

Enumerate $(\mathcal{A}^i)^\kappa$ for $i = 0, 1$ as

$$(\mathcal{A}^i)^\kappa = \{f^i_\alpha : \alpha < 2^\kappa\}.$$

By a back-and-forth method, we construct a sequence of triples $\langle(\mathcal{U}_\alpha, g^0_\alpha, g^1_\alpha) : \alpha < 2^\kappa\rangle$ satisfying:

(1) $g^0_\alpha \in (\mathcal{A}^0)^\kappa$,

(2) $g^1_\alpha \in (\mathcal{A}^1)^\kappa$,

(3) $\mathcal{U}_\alpha$ is a filter on κ generated by $\kappa + |\alpha|$ sets,

(4) $\langle \mathcal{U}_\alpha : \alpha < 2^\kappa\rangle$ is an increasing continuous sequence,

(5) If $\varphi(x_0, \dots, x_{n-1})$ is an $\mathcal{L}$-formula and $\beta_0, \dots, \beta_n \leq \alpha$, then the set

$$Y_{\varphi,\langle\beta_0,\dots,\beta_n\rangle} := \{\xi \in \kappa : \mathcal{A}^0 \models \varphi(g^0_{\beta_0}(\xi), \dots, g^0_{\beta_{n-1}}(\xi)) \iff \mathcal{A}^1 \models \varphi(g^1_{\beta_0}(\xi), \dots, g^1_{\beta_{n-1}}(\xi))\}$$

belongs to $\mathcal{U}_{\alpha+1}$.

In the construction, when α is even, we put $g^0_\alpha = f^0_\gamma$ where γ is the least ordinal $f^0_\gamma \notin \{g^0_\beta : \beta < \alpha\}$. And $\mathbb{P}$ is the poset of partial functions of size $<\kappa$ from κ to $\mathcal{A}^1$. This poset is forcing equivalent to $\mathrm{Fn}_\kappa(\kappa, 2)$.

Take a generating set $\mathcal{F}$ of $\mathcal{U}_\alpha$ of size $\aleph_0 + |\alpha|$. Then by using $\mathrm{MA}_{<2^\kappa}(\mathrm{Fn}_\kappa(\kappa, 2))$, take a $\mathbb{P}$-generic filter G with respect to the following family of dense sets of $\mathbb{P}$:

$$D_\xi = \{p \in \mathbb{P} : \xi \in \mathrm{dom}\, p\} \text{ (for } \xi \in \kappa)$$

and

$$\begin{aligned} E_{X,\langle\varphi_\iota:\iota\in I\rangle,\langle\gamma^\iota_1,\dots,\gamma^\iota_{n_\iota}:\iota\in I\rangle} = &\{p \in \mathbb{P} : (\exists \xi \in \mathrm{dom}(p) \cap X)(\forall \iota \in I) \\ &(\mathcal{A}^0 \models \varphi_\iota(g^0_{\gamma^\iota_1}(\xi), \dots g^0_{\gamma^\iota_{n_\iota}}(\xi), g^0_\alpha(\xi)) \Leftrightarrow \\ &\mathcal{A}^1 \models \varphi_\iota(g^1_{\gamma^\iota_1}(\xi), \dots g^1_{\gamma^\iota_{n_\iota}}(\xi), p(\xi))\}), \end{aligned}$$

where $X \in \mathcal{F}$, $\langle\varphi_\iota : \iota \in I\rangle$ is a finite sequence of $\mathcal{L}$-formulas and $\gamma^\iota_1, \dots, \gamma^\iota_{n_\iota}$ for $\iota \in I$ are ordinals less than α.

We now prove that $E := E_{X,\langle\varphi_\iota:\iota\in I\rangle,\langle\gamma^\iota_1,\dots,\gamma^\iota_{n_\iota}:\iota\in I\rangle}$ is dense. Let $p \in \mathbb{P}$. For each $\xi \in \kappa$ and $\iota \in I$, put

$$v(\xi, \iota) = \begin{cases} 1 & \text{if } \mathcal{A}^0 \models \varphi_\iota(g^0_{\gamma^\iota_1}(\xi), \dots, g^0_{\gamma^\iota_{n_\iota}}(\xi), g^0_\alpha(\xi)) \\ 0 & \text{otherwise.} \end{cases}$$

And for each $\xi \in \kappa$ put

$$v(\xi) = \langle v(\xi, \iota) : \iota \in I \rangle.$$

Then by finiteness of ${}^I 2$, for some $v_0 \in {}^I 2$, we have $\kappa \setminus v^{-1}(v_0) \notin \mathcal{U}_\alpha$.

For each $\iota \in I$, put

$$\varphi_\iota^+(x_1^\iota, \ldots, x_{n_\iota}^\iota, y) \equiv \begin{cases} \varphi_\iota(x_1^\iota, \ldots, x_{n_\iota}^\iota, y) & \text{if } v_0(\iota) = 1 \\ \neg\varphi_\iota(x_1^\iota, \ldots, x_{n_\iota}^\iota, y) & \text{otherwise.} \end{cases}$$

Put

$$\psi \equiv \exists y \bigwedge_{\iota \in I} \varphi_\iota^+(x_1^\iota, \ldots, x_{n_\iota}^\iota, y).$$

Then by the induction hypothesis (5), $Y_{\psi, \langle \gamma_1^\iota, \ldots \gamma_{n_\iota}^\iota : \iota \in I \rangle} \in \mathcal{U}_\alpha$. So take $\xi \in X \cap v^{-1}(v_0) \cap Y_{\psi, \langle \gamma_1^\iota, \ldots \gamma_{n_\iota}^\iota : \iota \in I \rangle} \setminus \mathrm{dom}(p)$.

Since $\mathcal{A}^0 \models \psi(\langle g^0_{\gamma_1^\iota}(\xi), \ldots g^0_{\gamma_{n_\iota}^\iota}(\xi) : \iota \in I \rangle)$, we have $\mathcal{A}^1 \models \psi(\langle g^1_{\gamma_1^\iota}(\xi), \ldots g^1_{\gamma_{n_\iota}^\iota}(\xi) : \iota \in I \rangle)$.

By the definition of ψ, we can take $y \in \mathcal{A}^1$ such that $\mathcal{A}^1 \models \varphi_\iota^+(g^1_{\gamma_1^\iota}(\xi), \ldots, g^1_{\gamma_{n_\iota}^\iota}(\xi), y)$ for every $\iota \in I$. We now put $q = p \cup \{(\xi, y)\}$. This witnesses denseness of E.

Then we put $g^1_\alpha = \bigcup G$ and letting $\mathcal{U}_{\alpha+1}$ contain $\mathcal{U}_\alpha$ and the sets in (5) and have either the α-th element of the enumeration of 2^κ or its complement.

When α is odd, do the same construction above except for swapping 0 and 1.

Then the construction guarantees that $\mathcal{U} = \bigcup_{\alpha < 2^\kappa} \mathcal{U}_\alpha$ is an ultrafilter and that the function

$$\langle ([g^0_\alpha]_\mathcal{U}, [g^1_\alpha]_\mathcal{U}) : \alpha < 2^\kappa \rangle$$

is an isomorphism from $(\mathcal{A}^0)^\mathcal{U}$ to $(\mathcal{A}^1)^\mathcal{U}$. $\square$

Fact 5.5.11 ([Vlu23, Theorem 4.3])**.** Let κ be an inaccessible cardinal. Then $\mathrm{cov}(\mathcal{M}_\kappa) \geq \lambda$ holds iff for every $X \subseteq \kappa^\kappa$ of size $<\lambda$ there is $S \in \prod_{i<\kappa} [\kappa]^{\leq |i|+1}$ such that for all $x \in X$ we have $\{i < \kappa : x(i) \in S(i)\}$ is cofinal in κ.

Fact 5.5.11 does not seem to generalize to anything other than inacessible cardinals. In fact, it is known that when κ is a successor cardinal, the cardinal invariant determined by slaloms as claimed above is equal to $\mathfrak{d}_\kappa$.

Theorem 5.5.12. Let κ be an inaccessible cardinal. Then $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$.

Proof. Let $\mathcal{U}$ be a regular ultrafilter on κ witnessing $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$. Let $X \subseteq \kappa^\kappa$ of size $<2^\kappa$. Define a language $\mathcal{L}$ by $\mathcal{L} = \{\subseteq\}$. For $i < \kappa$, define a $\mathcal{L}$-structure $\mathcal{A}_i$ by $\mathcal{A}_i = ([\kappa]^{<|i|}, \subseteq)$. Since κ is inaccessible, we have $|\mathcal{A}_i| = \kappa$. For $x \in \kappa^\kappa$, we define $S_x = \langle \{x(i)\} : i < \kappa \rangle$. Put $\mathcal{A}_* = \prod_{i<\kappa} \mathcal{A}_i / \mathcal{U}$. Consider a set of formulas $p(S)$ defined by

$$p(S) = \{\ulcorner [S_x] \subseteq S \urcorner : x \in X\}.$$

Then $p(S)$ is finitely satisfiable and the number of parameters occurring in $p(S)$ is $<2^\kappa$. Thus, by $\mathrm{SAT}_\kappa^{\aleph_0}(\kappa)$, we can take $[S] \in \mathcal{A}_*$ realizing $p(S)$. Then we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \in \mathcal{U}).$$

But since our ultrafilter $\mathcal{U}$ is uniform, we have

$$(\forall x \in X)(\{i < \kappa : x(i) \in S(i)\} \text{ is cofinal}).$$

So by Fact 5.5.11, we showed $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$. $\square$

Theorem 5.5.13. Let κ be a regular cardinal. Then $\mathrm{cov}(\mathcal{M}_\kappa) = 2^{<2^\kappa} = 2^\kappa$ implies $\mathrm{SAT}^\kappa_\kappa(\kappa)$.

Proof. Let $\langle b_\alpha : \alpha < 2^\kappa \rangle$ be an enumeration of κ^κ.

Let $\mathcal{L}^+ = \mathcal{L} \cup \{c_\alpha : \alpha < 2^\kappa\}$ where the c_α's are new constant symbols and let $\mathrm{Fml}(\mathcal{L}^+)$ be the set of all $\mathcal{L}^+$ formulas with one free variable.

Let $\langle (\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) : \xi < 2^\kappa \rangle$ be an enumeration of tuples $(\mathcal{L}, T, \mathcal{B}, \Delta)$ such that $\mathcal{L}$ is a language of size $\le \kappa$, $T \colon \kappa \to \kappa + 1$, $\mathcal{B} = \langle \mathcal{A}_i : i < \kappa \rangle$ is a κ-sequence of $\mathcal{L}$-structures with i-th universe $T(i)$ and Δ is a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Here we used $(2^\kappa)^{<2^\kappa} = 2^\kappa$. Ensure each $(\mathcal{L}, T, \mathcal{B}, \Delta)$ occurs cofinally in this sequence.

For $\mathcal{B}_\xi = \langle \mathcal{A}^\xi_i : i < \kappa \rangle$, we put

$$\mathcal{B}_\xi(i) = \langle \mathcal{A}^\xi_i, b_0(i) \restriction T_\xi(i), b_1(i) \restriction T_\xi(i), \dots \rangle,$$

which is a $\mathcal{L}^+$-structure. Here $\alpha \restriction \beta = \begin{cases} \alpha & \text{if } \alpha < \beta \\ 0 & \text{otherwise} \end{cases}$ for α and β are ordinals.

Let $\langle X_\xi : \xi < 2^\kappa \rangle$ be an enumeration of $\mathcal{P}(\kappa)$. We construct a sequence of filters $\langle F_\xi : \xi < 2^\kappa \rangle$ satisfying following conditions:

(1) F_0 is the filter generated by a regularizing set for κ.

(2) $F_\xi \subseteq F_{\xi+1}$ and $F_\xi = \bigcup_{\alpha<\xi} F_\alpha$ for a limit ξ.

(3) $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$.

(4) F_ξ is generated by $< 2^\kappa$ members.

(5) If

$$(\forall \Gamma \in [\Delta_\xi]^{<\aleph_0})(\{i < \kappa : \Gamma \text{ is satisfiable in } \mathcal{B}_\xi(i)\} \in F_\xi) \tag{$*$}$$

Then there is $f \in \prod_{i<\kappa} T_\xi(i)$ such that for every $\varphi \in \Delta_\xi$ we have $\{i < \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$.

Suppose that F_ξ is constructed and $(*)$ holds. Let

$$\mathbb{P} = \{p : p \text{ is a partial function of size} < \kappa \text{ from } \kappa \text{ to } \kappa\}$$

This forcing notion $\mathbb{P}$ is forcing equivalent to the forcing adding a κ-Cohen function.

Fix a generating set F'_ξ of F_ξ of size $< 2^\kappa$. For each $A \in F'_\xi$ and $\varphi_1, \dots, \varphi_n \in \Delta_\xi$, we put

$$\begin{aligned} E_{A,\varphi_1,\dots,\varphi_n} = \{p \in \mathbb{P} : (\exists i \in \mathrm{dom}(p) \cap A)(&p(i) \text{ is element of } T_\xi(i) \\ &\text{and satisfies } \varphi_1, \dots, \varphi_n \text{ in } \mathcal{B}_\xi(i))\} \end{aligned}$$

By assumption $(*)$, these $E_{A,\varphi_1,\dots,\varphi_n}$'s are dense subsets in $\mathbb{P}$.

So using $\mathrm{MA}_{<2^\kappa}(\mathbb{P})$, we have a filter G of $\mathbb{P}$ that intersects all $E_{A,\varphi_1,\dots,\varphi_n}$'s. Put $f(i) = (\bigcup G)(i) \restriction T_\xi(i)$. Then we can extend our filter F_ξ to $F_{\xi+1}$ such that for every $\phi \in \Delta_\xi$ $\{i < \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}_\xi(i)\} \in F_{\xi+1}$. Moreover we can extend this filter satisfying $X_\xi \in F_{\xi+1}$ or $\kappa \setminus X_\xi \in F_{\xi+1}$. This finishes the construction.

In order to check that the resulting ultrafilter $F = \bigcup_{\xi<2^\kappa} F_\xi$ witnesses $\mathrm{SAT}^\kappa_\kappa(\kappa)$, let $\mathcal{L}$ be a language of size $\leq \kappa$ and $\mathcal{B} = \langle \mathcal{A}_i : i \in \kappa \rangle$ be a sequence of $\mathcal{L}$-structures. We may assume that, for each $i < \kappa$, the universe of $\mathcal{A}_i$ is an ordinal. Let $T(i) =$ the universe of $\mathcal{A}_i$. Let Δ be a subset of $\mathrm{Fml}(\mathcal{L}^+)$ with $|\Delta| < 2^\kappa$. Assume that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma := \{i \in \kappa : \Gamma \text{ is satisfiable in } \mathcal{B}(i)\} \in F$. By the regularity of 2^κ which follows from the cardinal arithmetical assumption of the theorem, we have $\alpha < 2^\kappa$ such that for all $\Gamma \subseteq \Delta$ finite, $X_\Gamma \in F_\alpha$. Let $\xi \geq \alpha$ be satisfying $(\mathcal{L}_\xi, T_\xi, \mathcal{B}_\xi, \Delta_\xi) = (\mathcal{L}, T, \mathcal{B}, \Delta)$. Then by (5), there is a $f \in \prod_{i<\kappa} T(i)$ such that for all $\varphi \in \Delta$, $\{i \in \kappa : f(i) \text{ satisfies } \varphi \text{ in } \mathcal{B}(i)\} \in F$. Thus $\prod_{i\in\kappa} \mathcal{A}_i/F$ is saturated. $\square$

5.6 Open problems

The following three questions remain for the countable case.

Question 5.6.1. (1) Does $\mathrm{KT}(\aleph_1)$ imply a stronger hypothesis than $\mathfrak{mcf} = \aleph_1$? In particular does $\mathrm{KT}(\aleph_1)$ imply $\mathrm{non}(\mathcal{M}) = \aleph_1$?

(2) Does $\mathrm{KT}(\aleph_0)$ imply a stronger hypothesis than $\mathfrak{c}^\exists \leq \mathfrak{d}$? In particular does $\mathrm{KT}(\aleph_0)$ imply $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$?

(3) In the Sacks model, does $\mathrm{KT}(\aleph_0)$ hold? (If in this model $\neg\,\mathrm{KT}(\aleph_0)$ holds, we can separate $\mathrm{KT}(\aleph_0)$ and $\mathfrak{c}^\exists \leq \mathfrak{d}$.)

The following figure can be drawn for an inaccessible cardinal κ.

$$\begin{tikzcd}
2^\kappa = \kappa^+ \arrow[rrd] \arrow[dd, leftrightarrow] & & \\
& & \mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa = 2^{<2^\kappa} \arrow[d, leftrightarrow] \\
\mathrm{SAT}^\kappa_\kappa(2^\kappa) \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] & \mathrm{SAT}^\kappa_\kappa(\kappa^+) \arrow[r] \arrow[dd] & \mathrm{SAT}^\kappa_\kappa(\kappa) \arrow[d] \\
& & \mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa \arrow[d] \\
\mathrm{KT}^\kappa_\kappa(2^\kappa) \arrow[r] & \mathrm{KT}^\kappa_\kappa(\kappa^+) \arrow[r] \arrow[d] & \mathrm{KT}^\kappa_\kappa(\kappa) \arrow[d] \\
& \mathfrak{b}_\kappa = \kappa^+ & ???
\end{tikzcd}$$

In light of this, the following two questions naturally arise.

Question 5.6.2. (1) Can we eliminate the inaccessibility assumption from the result which states $\mathrm{SAT}^{\aleph_0}_\kappa(\kappa)$ implies $\mathrm{cov}(\mathcal{M}_\kappa) = 2^\kappa$?

(2) Can we prove the consistency of $\neg\,\mathrm{KT}^\kappa_\kappa(\kappa)$ for an uncountable cardinal κ?

As for the second item, we obtain the following.

Theorem 5.6.3. Let κ be an inaccessible cardinal. Then $\mathrm{KT}_\kappa^{\aleph_0}(\kappa)$ implies $\mathfrak{v}_\kappa^\forall \leq \mathfrak{d}_\kappa$. $\square$

Here, for a cardinal κ and $c, h \in \kappa^\kappa$, letting $\prod c = \prod_{\alpha<\kappa} c(\alpha)$ and $S(c,h) = \prod_{\alpha<\kappa}[c(\alpha)]^{<h(\alpha)}$, we define

$$\mathfrak{v}_{\kappa,c,h}^\forall = \min\{|X| : X \subseteq \prod c, (\forall \varphi \in S(c,h))(\exists x \in X)$$
$$(\forall \alpha < \kappa)(\exists \beta \in [\alpha,\kappa))(x(\alpha) \notin \varphi(\alpha))\}.$$

Also, we define $\mathfrak{v}_\kappa^\forall = \min\{\mathfrak{v}_{\kappa,c,h}^\forall : c, h \in \kappa^\kappa, \text{ and } h \text{ diverges to } \infty\}$.

However, for an inaccessible cardinal κ, the consistency of $\mathfrak{d}_\kappa < \mathfrak{v}_\kappa^\forall$ is not currently known. The situation differs from cardinal invariants at ω in that forcing notions such as random forcing are not known for higher cardinals, nor are good generalizations of properties such as ω^ω-bounding proper forcing.

Chapter 6

Comparability numbers and incomparability numbers

As cardinal invariants of a poset, the dominating number and the unbounding number are well-studied. In this chapter, as new cardinal invariants of a poset, we introduce the comparability number and incomparability number and determine their value for well-known posets.

Definition 6.0.1. Let $(P, \leq)$ be a poset. We say $F \subseteq P$ is a *dominating family* if for every $p \in P$ there is $q \in F$ such that $p \leq q$. We say $F \subseteq P$ is an *unbounded family* if for every $p \in P$ there is $q \in F$ such that $q \not\leq p$.

Define cardinal invariants $\mathfrak{d}(P, \leq)$ and $\mathfrak{b}(P, \leq)$ as follows:

(1) $\mathfrak{d}(P, \leq) = \min\{|F| : F \subseteq P \text{ dominating family}\}$,

(2) $\mathfrak{b}(P, \leq) = \min\{|F| : F \subseteq P \text{ unbounded family}\}$.

We call $\mathfrak{d}(P, \leq)$ the *dominating number* for P and $\mathfrak{b}(P, \leq)$ the *bounding number* for P.

Definition 6.0.2. Let $(P, \leq)$ be a poset. We say $F \subseteq P$ is a *comparable family* if for every $p \in P$ there is $q \in F$ such that either $p \leq q$ or $q \leq p$ holds. We say $F \subseteq P$ is an *incomparable family* if for every $p \in P$ there is $q \in F$ such that both $p \not\leq q$ and $q \not\leq p$ hold.

We define cardinal invariants $\mathfrak{cp}(P, \leq)$ and $\mathfrak{icp}(P, \leq)$ as follows:

(1) $\mathfrak{cp}(P, \leq) = \min\{|F| : F \subseteq P \text{ comparable family}\}$,

(2) $\mathfrak{icp}(P, \leq) = \min\{|F| : F \subseteq P \text{ incomparable family}\}$.

We call $\mathfrak{cp}(P, \leq)$ the *comparability number* for P and $\mathfrak{icp}(P, \leq)$ the *incomparability number* for P.

$\mathfrak{cp}(P)$ is always defined. On the other hand, $\mathfrak{icp}(P)$ may not be defined. $\mathfrak{icp}(P)$ is defined if and only if for all $p \in P$ there is $q \in P$ such that p and q are incomparable. This is equivalent to $\mathfrak{cp}(P) > 1$.

These cardinals are related to dominating numbers and bounding numbers: $\mathfrak{cp}(P) \leq \min\{\mathfrak{d}(P), \mathfrak{d}(P^*)\}$ and $\max\{\mathfrak{b}(P), \mathfrak{b}(P^*)\} \leq \mathfrak{icp}(P)$. Here, P^* is the poset with the reverse ordering of $(P, \leq)$.

As invariants related to comparability numbers and incomparability numbers, we can consider minimal sizes of maximal antichains and maximal chains.

Definition 6.0.3. Let $(P, \leq)$ be a poset. A subset $C \subseteq P$ is called a chain of P if members of C are pairwise comparable. Similarly, a subset $A \subseteq P$ is called an antichain of P if members of C are pairwise incomparable.

Define invariants $\mathfrak{mc}(P)$ and $\mathfrak{mac}(P)$ as follows:

(1) $\mathfrak{mc}(P) = \min\{|C| : C \subseteq P \text{ maximal chain}\}$, and

(2) $\mathfrak{mac}(P) = \min\{|A| : A \subseteq P \text{ maximal antichain}\}$.

As can be easily seen, a maximal antichain of P is a comparable family of P. So we have $\mathfrak{cp}(P) \leq \mathfrak{mac}(P)$. We can also observe that:

Lemma 6.0.4. If $\mathfrak{icp}(P)$ is defined and $\mathfrak{mc}(P)$ is infinite, then we have $\mathfrak{icp}(P) \leq \mathfrak{mc}(P)$.

Proof. Since $\mathfrak{icp}(P)$ is defined, for each $p \in P$, we can take $q_p \in P$ such that p and q_p are incomparable. Take a maximal chain C of P of size $\mathfrak{mc}(P)$. Then the set $C' := C \cup \{q_p : p \in C\}$ is clearly an incomparable family since C is maximal. The set C' has also size $\mathfrak{mc}(P)$ since it is infinite. $\square$

So we can draw a picture as in Figure 6.1 if $\mathfrak{icp}(P)$ is defined and $\mathfrak{mc}(P)$ is infinite.

$$\begin{tikzcd}
\mathfrak{d}(P) & \mathfrak{mac}(P) & \mathfrak{d}(P^*) \\
& \mathfrak{cp}(P) \arrow[ul] \arrow[u] \arrow[ur] & \\
& \mathfrak{mc}(P) & \\
& \mathfrak{icp}(P) \arrow[u] & \\
\mathfrak{b}(P) \arrow[uuuu] \arrow[ur] & & \mathfrak{b}(P^*) \arrow[uuuu] \arrow[ul]
\end{tikzcd}$$

Figure 6.1: Relationships

The results in Table 6.1 are well-known.

Table 6.2 summarizes almost all results we will prove in this paper.

As results not listed in the table, in Section 6.10, we treat ideals on ω, and in Section 6.11, we treat ideals on ω_1.

P	$\mathfrak{d}(P)$	$\mathfrak{b}(P)$	$\mathfrak{d}(P^*)$	$\mathfrak{b}(P^*)$
$(\omega^\omega \setminus \mathbb{0}, \leq^*)$	$\mathfrak{d}$	$\mathfrak{b}$	$\mathfrak{c}$	2
$(\mathcal{P}(\omega)/\mathsf{fin})^-$	$\mathfrak{c}$	2	$\mathfrak{c}$	2
$(\mathsf{Borel}(2^\omega)/\mathcal{M})^-$	$\aleph_0$	2	$\aleph_0$	2
$(\mathsf{Borel}(2^\omega)/\mathcal{N})^-$	$\mathrm{cof}(\mathcal{N})$	2	$\mathrm{cof}(\mathcal{N})$	2
$(\mathcal{N} \setminus \{\varnothing\}, \subseteq)$	$\mathrm{cof}(\mathcal{N})$	$\mathrm{add}(\mathcal{N})$	$\mathfrak{c}$	2
$(\mathcal{M} \setminus \{\varnothing\}, \subseteq)$	$\mathrm{cof}(\mathcal{M})$	$\mathrm{add}(\mathcal{M})$	$\mathfrak{c}$	2
the Turing degrees	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	2
$(\beta\omega \setminus \omega, \leq_{\mathrm{RK}})$	$2^{\mathfrak{c}}$	$\mathfrak{c}^+$	depends	depends

Table 6.1: Known results

P	$\mathfrak{cp}(P)$	$\mathfrak{icp}(P)$	$\mathfrak{mac}(P)$	$\mathfrak{mc}(P)$
$(\omega^\omega \setminus \mathbb{0}, \leq^*)$	$\mathfrak{d}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{c}$
$(\mathcal{P}(\omega)/\mathsf{fin})^-$	$\mathfrak{r}$	2	$\mathfrak{c}$ [1]	$\mathfrak{c}$
$(\mathsf{Borel}(2^\omega)/\mathcal{M})^-$	$\aleph_0$	2	?	$\mathfrak{c}$
$(\mathsf{Borel}(2^\omega)/\mathcal{N})^-$	$\mathrm{cof}(\mathcal{N})$	2	?	$\mathfrak{c}$
$(\mathcal{N} \setminus \{\varnothing\}, \subseteq)$	$\mathrm{cof}(\mathcal{N})$	$\mathrm{add}(\mathcal{N})$	$\mathfrak{c}$	$\mathrm{non}(\mathcal{N})$
$(\mathcal{M} \setminus \{\varnothing\}, \subseteq)$	$\mathrm{cof}(\mathcal{M})$	$\mathrm{add}(\mathcal{M})$	$\mathfrak{c}$	$\mathrm{non}(\mathcal{M})$
the Turing degrees	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$(\beta\omega \setminus \omega, \leq_{\mathrm{RK}})$	depends	$\mathfrak{c}^+$ or undefined	?	$\mathfrak{c}^+$

Table 6.2: Our results

Finally, we give an example of a poset with small comparability number. Let $P = \{0,1\} \times \mathbb{Z}$ and order P by

$$(i,m) \leq (j,n) \iff (i = j \land m \leq n) \lor (i \neq j \land m < n).$$

Then, since $\{(0,0),(1,0)\}$ is a maximal antichain, we have $\mathfrak{mac}(P) = \mathfrak{cp}(P) = 2$. On the other hand, we have $\mathfrak{d}(P) = \mathfrak{d}(P^*) = \mathfrak{b}(P) = \mathfrak{b}(P^*) = \mathfrak{icp}(P) = \aleph_0$.

6.1 General lemmas

The following 3 lemmas are well known and easy to see.

Lemma 6.1.1. Let P be a poset. Suppose that P has the following property:

$$\text{If } a < b \text{ in } P \text{ then there is } c \in P \text{ such that } a < c < b. \tag{$*$}$$

Then P embeds the set of rational numbers $\mathbb{Q}$.

Lemma 6.1.2. Let P be a poset. Assume P has the property in Lemma 6.1.1. Moreover, suppose that P has the following property:

$$\begin{aligned} &\text{If } \langle a_n : n \in \omega\rangle \text{ is an increasing sequence of } P \text{ and} \\ &\quad \langle b_m : m \in \omega\rangle \text{ is a decreasing sequence of } P \text{ and} \\ &\quad\quad (\forall n, m \in \omega)(a_n < b_m) \text{ holds,} \\ &\quad\quad\quad \text{then there is } c \in P \text{ such that } (\forall n, m \in \omega)(a_n < c < b_m). \end{aligned} \tag{$**$}$$

Then P embeds the set of real numbers $\mathbb{R}$.

Lemma 6.1.3. Both $(*)$ and $(**)$ in Lemma 6.1.1 and 6.1.2 are inherited by any maximal chains.

6.2 The cardinal invariants of ω^ω

In this section, we determine the comparability number and the incomparability number of ω^ω as a first result.

[1]This result was obtained by [CCHM16]

Definition 6.2.1. Let $\mathbb{0}$ be the set of eventually zero reals, that is,

$$\mathbb{0} = \{x \in \omega^\omega : (\forall^\infty n)(x(n) = 0)\}.$$

We consider the poset $(\omega^\omega \setminus \mathbb{0}, \leq^*)$.

Lemma 6.2.2. $\mathfrak{b} \leq \mathfrak{icp}(\omega^\omega \setminus \mathbb{0})$ and $\mathfrak{cp}(\omega^\omega \setminus \mathbb{0}) \leq \mathfrak{d}$ hold.

Proof. This is immediate from the definition. $\square$

The proofs of the following two propositions (Proposition 6.2.3 and 6.2.4) are suggested by an anonymous reviewer.

Proposition 6.2.3. $\mathfrak{icp}(\omega^\omega \setminus \mathbb{0}) \leq \mathfrak{b}$ holds.

Proof. Take an unbounded family $F \subseteq \omega^\omega \setminus \mathbb{0}$ of size $\mathfrak{b}$. For $f \in F$, we define $f_\mathrm{e}, f_\mathrm{o} \in \omega^\omega$ as follows:

$$f_\mathrm{e}(n) = \begin{cases} f(n/2) & \text{(if } n \text{ is even)} \\ 0 & \text{(if } n \text{ is odd)} \end{cases}$$

$$f_\mathrm{o}(n) = \begin{cases} 0 & \text{(if } n \text{ is even)} \\ f((n-1)/2) & \text{(if } n \text{ is odd)} \end{cases}$$

Then the set $\{f_\mathrm{e} : f \in F\} \cup \{f_\mathrm{o} : f \in F\}$ is an incomparable family. To see it, fix $g \in \omega^\omega \setminus \mathbb{0}$. Then the set $\{n \in \omega : g(n) > 0\}$ is infinite. So either $\{n \text{ even number} : g(n) > 0\}$ or $\{n \text{ odd number} : g(n) > 0\}$ is infinite. If $\{n \text{ even number} : g(n) > 0\}$ is infinite, then there is $f \in F$ such that $\langle g(2n+1) : n \in \omega\rangle <^\infty f$. We can deduce from it that g and f_o are incomparable. In the case $\{n \text{ odd number} : g(n) > 0\}$ is infinite, a similar proof can be done. $\square$

Proposition 6.2.4. $\mathfrak{d} \leq \mathfrak{cp}(\omega^\omega \setminus \mathbb{0})$ holds.

Proof. Take a comparable family C of size $\mathfrak{cp}(\omega^\omega \setminus \mathbb{0})$. We produce a dominating family D such that $|D| \leq |C|$. If $|C| = \mathfrak{c}$, then such D exists obviously. So we can assume that $|C| < \mathfrak{c}$.

Fix an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$. Since $|C| < |\mathcal{A}|$ and $\mathcal{A}$ is almost disjoint, we can take $A \in \mathcal{A}$ such that for all $f \in C$, we have $\neg(\{n : f(n) > 0\} \subseteq^* A)$. Note that $f \leq^* g$ in ω^ω implies $\{n : f(n) > 0\} \subseteq^* \{n : g(n) > 0\}$. So for every $g \in \omega^\omega$ with $\{n : g(n) > 0\} = A$ and $f \in C$, we have $\neg(f \leq^* g)$. But since C is a comparable family, for every $g \in \omega^\omega$ with $\{n : g(n) > 0\} = A$ there is $f \in C$ such that $g \leq^* f$.

Let $\pi \colon \omega \to A$ be a bijection. The observation in the previous paragraph implies

$$D = \{f \circ \pi : f \in C\}$$

is a dominating family. Since $|D| \leq |C|$, we are done. $\square$

Thus, we have $\mathfrak{icp}(\omega^\omega \setminus \mathbb{0}) = \mathfrak{b}$ and $\mathfrak{cp}(\omega^\omega \setminus \mathbb{0}) = \mathfrak{d}$, but it is natural to ask whether these can be shown by Tukey reducibility. Theorem 6.2.6 below answers this.

Definition 6.2.5. Define a relational system **ICP** as follows:

(1) $\mathbf{ICP} = (\omega^\omega \setminus \mathbb{0}, \omega^\omega \setminus \mathbb{0}, <^\infty \cap >^\infty)$.

Theorem 6.2.6. There is a Tukey morphism from **ICP** to $\mathbf{B}^{\mathrm{IP}}$.

Proof. We use the notation j_k for the minimum number of k-th interval in an interval partition $\langle J_k : k \in \omega\rangle$.

We have to construct maps $\varphi\colon \omega^\omega \setminus \mathbb{0} \to \mathsf{IP}$ and $\psi\colon \mathsf{IP} \to \omega^\omega \setminus \mathbb{0}$ that satisfy the following condition:

$$\text{If } x \in \omega^\omega \setminus \mathbb{0},\ \mathbb{J} = \langle J_k : k \in \omega\rangle \in \mathsf{IP} \text{ satisfy}$$
$$(\exists^\infty n)(\forall k)(J_k \not\subseteq \varphi(x)(n)) \text{ then } x <^\infty \psi(\mathbb{J}) \text{ and } x >^\infty \psi(\mathbb{J}).$$

Enumerate $\{n : x(n) > 0\}$ by $\{n : x(n) > 0\} = \{a^x_0 < a^x_1 < a^x_2 < \dots\}$. Define φ and ψ by the following way:

$$\varphi(x)(n) = [i_n, i_{n+1}),$$

where $i_0 = 0$ and i_{n+1} are such that the interval $[i_n, i_{n+1})$ contains at least 3 points of the form a^x_j and for all $a \le i_n$, $x(a) \le i_{n+1}$ and

$$\psi(\mathbb{J})(n) = \begin{cases} \min J_{k+2} & (\text{if } n \in J_k \text{ and } n = \min J_k) \\ 0 & (\text{if } n \in J_k \text{ and } n > \min J_k). \end{cases}$$

We first show that $x >^\infty \psi(\mathbb{J})$. Take $n_0 \in \omega$ arbitrarily. Then we can take $n > n_0$ such that $(\forall k)(J_k \not\subseteq \varphi(x)(n))$. Let $I_n = \varphi(x)(n)$. Then we take k such that $I_n \cap J_k \neq \varnothing$. Note that the number of such k is less than or equal to 2. But we have at least 3 points a^x_i in I_n. So we can take $a^x_i \in I_n$ that is not the leftmost point of intervals in $\mathbb{J}$. We have $a^x_i \ge a^x_{3n} \ge 3n > n_0$, $x(a^x_i) > 0$ and $\psi(\mathbb{J})(a^x_i) = 0$. Thus we have $x >^\infty \psi(\mathbb{J})$.

We next prove $x <^\infty \psi(\mathbb{J})$. Let $k_0 \in \omega$. By $(\exists^\infty n)(\forall k)(J_k \not\subseteq \varphi(x)(n))$, we can take n such that $i_n > j_{k_0}$ and $(\forall k)(J_k \not\subseteq I_n)$. Let k be such that $i_n \in J_k$. Then $j_k \le i_n$ and $i_{n+1} < j_{k+2}$ since there are at most 2 intervals in $\mathbb{J}$ touching I_n. By the choice of i_{n+1}, we have $x(j_k) \le i_{n+1} < j_{k+2}$. Thus $x(j_k) < \psi(\mathbb{J})(j_k)$. Also, by $i_n \in J_k$, we have $i_n < j_{k+1}$. So $j_{k_0} < i_n < j_{k+1}$. Thus $k_0 \le k$. Thus we have proved $x <^\infty \psi(\mathbb{J})$. □

Theorem 6.2.7. $\mathfrak{mc}(\omega^\omega \setminus \mathbb{0}) = \mathfrak{c}$.

Proof. Every maximal chain of $\omega^\omega \setminus \mathbb{0}$ satisfies the assumption in Lemma 6.1.2. □

The following theorem was obtained through private communication with Jorge Antonio Cruz Chapital.

Theorem 6.2.8. $\mathfrak{mac}(\omega^\omega \setminus \mathbb{0}) = \mathfrak{c}$.

Proof. Let $\mathcal{A}$ be a maximal antichain of $\omega^\omega \setminus \mathbb{0}$. Fix $\psi \in \mathcal{A}$. Let $X = \{n \in \omega : \psi(n) > 0\}$. Take a family $\langle (A_\alpha, B_\alpha) : \alpha < \mathfrak{c}\rangle$ of pairs of elements in $[X]^\omega$ such that $A_\alpha \cap B_\alpha = \varnothing$ for every α and $A_\alpha \cup B_\alpha$ and $A_\beta \cup B_\beta$ are almost disjoint for every distinct α and β. For $\alpha < \mathfrak{c}$, we define g_α by

$$g_\alpha(n) = \begin{cases} \psi(n) + 1 & (\text{if } n \in A_\alpha) \\ \psi(n) - 1 & (\text{if } n \in B_\alpha) \\ \psi(n) & (\text{otherwise}). \end{cases}$$

Define two sets $Y_0, Y_1 \subseteq \mathfrak{c}$ by

$$Y_0 = \{\alpha < \mathfrak{c} : (\exists f \in \mathcal{A})(g_\alpha \leq^* f)\}$$
$$Y_1 = \{\alpha < \mathfrak{c} : (\exists f \in \mathcal{A})(f \leq^* g_\alpha)\}$$

Since $Y_0 \cup Y_1 = \mathfrak{c}$, we have either $|Y_0| = \mathfrak{c}$ or $|Y_1| = \mathfrak{c}$.

Consider the case $|Y_0| = \mathfrak{c}$. For each $\alpha \in Y_0$, take $f_\alpha \in \mathcal{A}$ such that $g_\alpha \leq^* f_\alpha$. Then for each $\alpha \in Y_0$, we have $\{n : f_\alpha(n) < \psi(n)\} \subseteq^* B_\alpha$. Note that $\{n : f_\alpha(n) < \psi(n)\}$ is an infinite set since f_α and ψ are distinct elements of $\mathcal{A}$. Therefore, for distinct α and β, we have $\{n : f_\alpha(n) < \psi(n)\}$ and $\{n : f_\beta(n) < \psi(n)\}$ are almost disjoint. Thus, we have proved $f_\alpha \neq f_\beta$ whenever α and β are distinct. So it holds that $|\mathcal{A}| = \mathfrak{c}$.

The proof is similar for the case $|Y_1| = \mathfrak{c}$. □

6.3 The cardinal invariants of Boolean algebras

In this section, we deal with (in)comparability numbers of Boolean algebras. We write the Boolean operations as $+, \cdot$ and $(-)^{\mathrm{c}}$: join, meet and complementation. Moreover, 0 and 1 mean the minimum and maximum elements of the Boolean algebra.

Definition 6.3.1. Let B be a Boolean algebra. Then we define B^- by

$$B^- = B \smallsetminus \{0, 1\}.$$

Lemma 6.3.2. Let B be a Boolean algebra that is not equal to $\{0, 1\}$. Then $\mathfrak{icp}(B^-) = 2$.

Proof. Take an element $b \in B \smallsetminus \{0, 1\}$. Then $F = \{b, b^{\mathrm{c}}\}$ satisfies

$$(\forall x \in B^-)(\exists y \in F)(x \not\leq y \,\&\, y \not\leq x).$$

In order to show this, let $x \in B^-$. Assume that $x \leq b$ or $b \leq x$. In either case, we can easily show that both $x \not\leq b^{\mathrm{c}}$ and $b^{\mathrm{c}} \not\leq x$. □

Definition 6.3.3. Let B be a Boolean algebra and D be a subset of $B \smallsetminus \{0\}$. We say D is *weakly dense set* of B if for all $b \in B \smallsetminus \{0\}$ there is $d \in D$ such that $d \leq b$ or $d \leq b^{\mathrm{c}}$. Put

$$\mathbf{wd}(B) = \min\{|D| : D \text{ is weakly dense set of } B\}$$

Lemma 6.3.4. If B is an atomless Boolean algebra, then $\mathbf{wd}(B)$ is infinite.

Proof. Suppose that D is a finite weakly dense set. Let D' be the set of finite meets of elements of D that is not equal to 0. Let D'' be the set of minimal elements of D'. Then D'' is a finite weakly dense set such that for every distinct $d, e \in D''$, we have $d \cdot e = 0$. We may assume that the given D has this property.

Enumerate D as $D = \{d_0, \ldots, d_{n-1}\}$. For each $i < n$, take an element e_i such that $0 < e_i < d_i$. We can take these elements since B is atomless. Put $b = e_0 + \cdots + e_{n-1}$. Then we have $d_i \not\leq b$ and $d_i \not\leq b^{\mathrm{c}}$ for every $i < n$. This is a contradiction. □

Lemma 6.3.5. Let B be a Boolean algebra. Then we have $\mathfrak{cp}(B^-) \le 2\mathbf{wd}(B)$ and $\mathbf{wd}(B) \le 2\mathfrak{cp}(B^-)$. In particular, if either $\mathfrak{cp}(B^-)$ or $\mathbf{wd}(B)$ is infinite, then we have $\mathfrak{cp}(B^-) = \mathbf{wd}(B)$.

Proof. First we show $\mathbf{wd}(B) \le 2\mathfrak{cp}(B^-)$. Let C be a comparable family of B^- of size $\mathfrak{cp}(B^-)$. Then $C' = C \cup \{c^c : c \in C\}$ is a weakly dense set of B. Now we have $|C'| \le 2|C| = 2\mathfrak{cp}(B^-)$. So $\mathbf{wd}(B) \le 2\mathfrak{cp}(B^-)$.

Next we show $\mathfrak{cp}(B^-) \le 2\mathbf{wd}(B)$. Let D be a weakly dense family of B of size $\mathbf{wd}(B^-)$. Then $D' = D \cup \{d^c : d \in D\}$ is a comparable family of B^-. Now we have $|D'| \le 2|D| = 2\mathbf{wd}(B)$. So $\mathfrak{cp}(B^-) \le 2\mathbf{wd}(B)$. □

6.4 The cardinal invariants of $\mathcal{P}(\omega)/\mathsf{fin}$

Corollary 6.4.1. $\mathfrak{cp}((\mathcal{P}(\omega))/\mathrm{fin})^-) = \mathfrak{r}$. □

Proof. This follows from Lemma 6.3.5. □

The following fact was discovered by G. Campero-Arena, J. Cancino, M. Hrušák and F. E. Miranda-Perea.

Fact 6.4.2 ([CCHM16, Corollary 2.4])**.** $\mathfrak{mac}((\mathcal{P}(\omega)/\mathsf{fin})^-) = \mathfrak{c}$.

6.5 The cardinal invariants of the Cohen algebra and the random algebra

Corollary 6.5.1. $\mathfrak{cp}((\mathsf{Borel}(2^\omega)/\mathcal{N})^-) = \mathrm{cof}(\mathcal{N})$.

Proof. This follows from Lemma 6.3.5 and Theorem 1 in [Bur89] that states that $\mathbf{wd}(\mathsf{Borel}(2^\omega)/\mathcal{N}) = \mathrm{cof}(\mathcal{N})$. □

Proposition 6.5.2. $\mathfrak{mc}((\mathsf{Borel}(2^\omega)/\mathcal{N})^-) = \mathfrak{mc}((\mathsf{Borel}(2^\omega)/\mathcal{M})^-) = \mathfrak{c}$.

Proof. This follows from the fact that the above two Boolean algebras are σ-complete and lemmas in Section 6.1. □

6.6 The cardinal invariants of the ideal $\mathcal{N}$

In this section, we determine the values $\mathfrak{cp}(\mathcal{N} \setminus \{\varnothing\})$ and $\mathfrak{icp}(\mathcal{N} \setminus \{\varnothing\})$.

Fact 6.6.1 ([BJ95, Lemma 1.3.23])**.** Suppose that $\langle a_n : n \in \omega \rangle$ is a sequence of reals in $(0, 1)$. Then there is a sequence $\langle A_n : n \in \omega \rangle$ of open sets of 2^ω such that it is independent in the sense of probability theory and $\mu(A_n) = a_n$.

Lemma 6.6.2. If $\mathcal{F} \subseteq \mathcal{N}$ is a family of size less than $\mathrm{cof}(\mathcal{N})$, then there is a $B \in \mathcal{N}$ such that for all $A \in \mathcal{F}$ we have $|B \setminus A| = \mathfrak{c}$.

Proof. This proof is based on [BJ95, Lemma 2.3.3]. Let $\mathcal{C} = \{S \in (\omega^{<\omega})^\omega : \sum \frac{|S(n)|}{(n+1)^2} < \infty\}$. And for $S, S' \in \mathcal{C}$, define $S \le S'$ by $S \le S' \iff (\forall^\infty)(S(n) \subseteq S'(n))$. It is known that $\mathcal{C}$ and $\mathcal{N}$ are Tukey equivalent. So it suffices to show that $\mathcal{C} \le_{\mathrm{T}} (\mathcal{N}, \mathcal{N}, \subseteq^*)$. Here $A \subseteq^* B$ means that $|A \setminus B| < \mathfrak{c}$.

We have to construct φ, ψ such that $\varphi\colon \mathcal{C} \to \mathcal{N}$, $\psi\colon \mathcal{N} \to \mathcal{C}$ and $(\forall S \in \mathcal{C})(\forall G \in \mathcal{N})(\varphi(S) \subseteq^* G \to S \leq \psi(G))$ hold.

By Fact 6.6.1, fix a sequence $(G_{n,i} : n, i \in \omega)$ of open sets such that $G_{n,i}$ has measure $1/(n+1)^2$ and the sequence $(G_{n,i} : n, i \in \omega)$ is independent.

Define $\varphi\colon \mathcal{C} \to \mathcal{N}$ by

$$\varphi(S) = \bigcap_{m\in\omega} \bigcup_{n\geq m} \bigcup_{i\in S(n)} G_{n,i}.$$

For $G \in \mathcal{N}$, fix a perfect set K^G of positive measure such that $G \cap K^G = \varnothing$. We can assume that $K^G \cap U \neq \varnothing$ implies $\mu(K^G \cap U) > 0$ for every basic open set U. Let $(U_n : n \in \omega)$ be an enumeration of all basic open sets U such that $K^G \cap U \neq \varnothing$. Put

$$A^G_{n,i} = \{j \in \omega : K^G \cap U_n \cap G_{i,j} = \varnothing\}.$$

Then we have

$$0 < \mu(K^G \cap U_n) \leq \mu\left(\bigcap_{i\in\omega} \bigcap_{j\in A^G_{n,i}} 2^\omega \setminus G_{i,j}\right) = \prod_{i\in\omega} \prod_{j\in A^G_{n,i}} \mu(2^\omega \setminus G_{i,j}).$$

So we have

$$0 < \prod_{i\in\omega} \left(1 - \frac{1}{(i+1)^2}\right)^{|A^G_{n,i}|}.$$

So by the relationship between convergence of infinite sums and that of infinite products, we have

$$\sum_{i\in\omega} \frac{|A^G_{n,i}|}{(i+1)^2} < \infty.$$

Therefore, we showed that $A^G_{n,i} \in \mathcal{C}$.

Take a slalom $S \in \mathcal{C}$ such that $(A^G_{n,i} : i \in \omega) \leq S$ for all $n \in \omega$. Define $\psi(G)$ by letting $\psi(G)$ be this S.

We have to show $(\forall S \in \mathcal{C})(\forall G \in \mathcal{N})(\varphi(S) \subseteq^* G \to S \leq \psi(G))$. Fix $S \in \mathcal{C}$ and $G \in \mathcal{N}$. Then we have $|\varphi(S) \cap K^G| \leq |\varphi(S) \setminus G| < \mathfrak{c}$. Since $\varphi(S) \cap K_G$ is a Borel set, we have $|\varphi(S) \cap K^G| \leq \aleph_0$ by the perfect set theorem.

We have

$$\bigcap_{m\in\omega} (K^G \cap \bigcup_{n\geq m} \bigcup_{i\in S(n)} G_{n,i}) \cap \bigcap_{x\in\varphi(S)\cap K^G} (K^G \setminus \{x\}) = \varnothing.$$

So by the Baire category theorem applied to the space K^G, at least one term in the above intersection is not dense in K^G. So, there is a $n_0 \in \omega$ such that $K^G \cap \bigcup_{n\geq n_0} \bigcup_{i\in S(n)} G_{n,i}$ is not dense in K^G. So we can take $m \in \omega$ such that $K^G \cap U_m \cap \bigcup_{n\geq n_0} \bigcup_{i\in S(n)} G_{n,i} = \varnothing$. Then we have $(\forall n \geq n_0)(\forall i \in S(n))(K^G \cap U_m \cap G_{n,i} = \varnothing)$. So we have $(\forall^\infty n)(S(n) \subseteq A^G_{m,n} \subseteq \psi(G)(n))$. Thus $S \leq \psi(G)$ holds. $\square$

Theorem 6.6.3. $\mathfrak{cp}(\mathcal{N} \setminus \{\varnothing\}) = \mathrm{cof}(\mathcal{N})$.

Proof. It is clear that $\mathfrak{cp}(\mathcal{N} \setminus \{\varnothing\}) \leq \mathrm{cof}(\mathcal{N})$. So it suffices to show $\mathrm{cof}(\mathcal{N}) \leq \mathfrak{cp}(\mathcal{N} \setminus \{\varnothing\})$.

Suppose $\kappa < \mathrm{cof}(\mathcal{N})$ and take $\mathcal{F} \subseteq \mathcal{N} \setminus \{\varnothing\}$ of size κ. Then by Lemma 6.6.2, we can take $B \in \mathcal{N}$ such that for all $A \in \mathcal{F}$ we have $|B \setminus A| = \mathfrak{c}$. For each $A \in \mathcal{F}$, fix an element $x_A \in A$. Put

$B' = B \smallsetminus \{x_A : A \in \mathcal{F}\}$. Then B' is a incomparable with all $A \in \mathcal{F}$, since $x_A \in A \smallsetminus B'$ and $|B \smallsetminus A| = \mathfrak{c}$ and $|B \smallsetminus B'| < \mathfrak{c}$. □

Theorem 6.6.4. $\mathfrak{icp}(\mathcal{N} \smallsetminus \{\varnothing\}) = \mathrm{add}(\mathcal{N})$.

Proof. It is clear that $\mathrm{add}(\mathcal{N}) \leq \mathfrak{icp}(\mathcal{N} \smallsetminus \{\varnothing\})$. So we have to show that $\mathfrak{icp}(\mathcal{N} \smallsetminus \{\varnothing\}) \leq \mathrm{add}(\mathcal{N})$. Take a sequence $\langle A_\alpha : \alpha < \mathrm{add}(\mathcal{N})\rangle$ of null sets whose union is not null. Put $B_\alpha = A_\alpha \smallsetminus \bigcup_{\beta<\alpha} A_\beta$. Then $\mathcal{F} = \{B_\alpha : \alpha < \mathrm{add}(\mathcal{N})\} \smallsetminus \{\varnothing\}$ is an incomparable family. To prove this, let $C \in \mathcal{N} \smallsetminus \{\varnothing\}$. Since we have $C \in \mathcal{N}$ and $\bigcup \mathcal{F} \notin \mathcal{N}$, there is an $\alpha < \mathrm{add}(\mathcal{N})$ such that $B_\alpha \not\subseteq C$. If $C \not\subseteq B_\alpha$ holds, then we are done. If $C \subseteq B_\alpha$ holds, then we take another piece B_β. Then C and B_β are disjoint nonempty sets, in particular, they are incomparable. □

Proposition 6.6.5. $\mathfrak{mc}(\mathcal{N}) = \mathrm{non}(\mathcal{N})$.

Proof. We first prove $\mathfrak{mc}(\mathcal{N}) \leq \mathrm{non}(\mathcal{N})$. Take a non-null set $X = \{x_\alpha : \alpha < \mathrm{non}(\mathcal{N})\}$. For each α, set $X_\alpha = \{x_\beta : \beta < \alpha\}$. Then $\{X_\alpha : \alpha < \mathrm{non}(\mathcal{N})\}$ is a maximal chain.

We next prove $\mathrm{non}(\mathcal{N}) \leq \mathfrak{mc}(\mathcal{N})$. Take a maximal chain $\mathcal{C}$ of $\mathcal{N}$. We have $\bigcup \mathcal{C} \notin \mathcal{N}$. In fact, otherwise, we can extend the chain $\mathcal{C}$ upwards. Set $X = \bigcup \mathcal{C}$.

For each $x \in X$, put

$$\mathcal{L}_x = \{C \in \mathcal{C} : x \notin C\},$$
$$\mathcal{R}_x = \{D \in \mathcal{C} : x \in D\}.$$

Then we have $\mathcal{L}_x \cup \mathcal{R}_x = \mathcal{C}$ (disjoint union) and for every $C \in \mathcal{L}_x$ and $D \in \mathcal{R}_x$, $C \subseteq D$. We put $D_x = \bigcap \mathcal{R}_x$. By maximality of $\mathcal{C}$, we have $D_x \in \mathcal{C}$. In addition, it can be easily shown that the map $X \ni x \mapsto D_x \in \mathcal{C}$ is injective.

Therefore, we have $\mathrm{non}(\mathcal{N}) \leq |X| \leq |\mathcal{C}|$. So it holds that $\mathrm{non}(\mathcal{N}) \leq \mathfrak{mc}(\mathcal{N})$. □

Proposition 6.6.6. $\mathfrak{mac}(\mathcal{N} \smallsetminus \{\varnothing\}) = \mathfrak{c}$.

Proof. This proof is based on [CCHM16, Proposition 2.3]. Clearly, $\{\{x\} : x \in 2^\omega\}$ is a maximal antichain of $\mathcal{N} \smallsetminus \{\varnothing\}$. So we have $\mathfrak{mac}(\mathcal{N} \smallsetminus \{\varnothing\}) \leq \mathfrak{c}$.

Let $A, A' \in \mathcal{N}$ be such that $|A| = |A'| = \mathfrak{c}$ and $A \cap A' = \varnothing$. To prove $\mathfrak{mac}(\mathcal{N} \smallsetminus \{\varnothing\}) \geq \mathfrak{c}$, let $\mathcal{A}$ be an antichain of size $<\mathfrak{c}$. Let $\mathcal{C}$ be the closure of $\mathcal{A} \cup \{A, A'\}$ under the operation of finite unions, finite intersections and taking difference sets. Since we have $|\mathcal{C}| < \mathfrak{c}$, which is the density of each of $\mathcal{P}(A) \smallsetminus \{\varnothing\}$ and $\mathcal{P}(A') \smallsetminus \{\varnothing\}$, we can take $C_0 \subseteq A'$ and $C_1 \subseteq A$ nonempty such that

$$\neg(\exists B \in \mathcal{C} \smallsetminus \{\varnothing\})(B \subseteq C_0 \text{ or } B \subseteq C_1). \tag{$*$}$$

Set $D = (A \smallsetminus C_1) \cup C_0$.

We claim $D \notin \mathcal{A}$. If $D \in \mathcal{A}$ holds, then we have $D \smallsetminus A = C_0 \in \mathcal{C} \smallsetminus \{\varnothing\}$, which contradicts $(*)$. Fix $X \in \mathcal{A}$ arbitrary. We next claim D and X are incomparable. If $D \subseteq X$, then $A \smallsetminus X \subseteq A \smallsetminus D = C_1$ holds. This contradicts $A \smallsetminus X \in \mathcal{C} \smallsetminus \{\varnothing\}$ and $(*)$. If $X \subseteq D$, then $X \smallsetminus A \subseteq D \smallsetminus A = C_0$ holds. This contradicts $X \smallsetminus A \in \mathcal{C} \smallsetminus \{\varnothing\}$ and $(*)$.

Therefore, we have $\mathcal{A} \cup \{D\}$ is bigger antichain than $\mathcal{A}$. So $\mathcal{A}$ is not maximal. □

6.7 The cardinal invariants of the ideal $\mathcal{M}$

In this section, we determine the values $\mathfrak{cp}(\mathcal{M} \setminus \{\varnothing\})$ and $\mathfrak{icp}(\mathcal{M} \setminus \{\varnothing\})$ by the same method as in the previous section.

Definition 6.7.1. For an interval partition $\mathbb{I} = (I_n : n \in \omega)$ and a real $x \in 2^\omega$, we put

$$\mathrm{Match}(x, \mathbb{I}) = \{y \in 2^\omega : (\exists^\infty n)(x \upharpoonright I_n = y \upharpoonright I_n)\}.$$

Fact 6.7.2. (1) $\mathrm{Match}(x, \mathbb{I})$ is a comeager set for every interval partition $\mathbb{I} = (I_n : n \in \omega)$ and every real $x \in 2^\omega$.

(2) [Bla10, Theorem 5.2] For every meager set $A \subseteq 2^\omega$, there is an interval partition $\mathbb{I} = (I_n : n \in \omega)$ and a real $x \in 2^\omega$ such that $A \cap \mathrm{Match}(x, \mathbb{I}) = \varnothing$.

Lemma 6.7.3. Let $\mathbb{I} = (I_n : n \in \omega), \mathbb{J} = (J_k : k \in \omega) \in \mathsf{IP}$ and $x, y \in 2^\omega$. Suppose that $|J_k| \geq 2$ for every k. Then the following are equivalent.

(1) $\mathrm{Match}(x, \mathbb{I}) \not\subseteq \mathrm{Match}(y, \mathbb{J})$.

(2) The set $\mathrm{Match}(x, \mathbb{I}) \setminus \mathrm{Match}(y, \mathbb{J})$ has size $\mathfrak{c}$.

(3) $(\exists^\infty n)(\forall k)(J_k \not\subseteq I_n \text{ or } x \upharpoonright J_k \neq y \upharpoonright J_k)$

Proof. This lemma is an improvement of [Bla10, Proposition 5.3]. That (2) implies (1) is clear. Moreover, that (1) implies (3) is not difficult. So we shall show (3) implies (2). Take an infinite set $A \subseteq \omega$ such that

$$(\forall n \in A)(\forall k)(J_k \not\subseteq I_n \text{ or } x \upharpoonright J_k \neq y \upharpoonright J_k). \tag{$*$}$$

We can assume that

$$(\forall n)(\{n, n+1\} \not\subseteq A). \tag{$**$}$$

Let

$$A' = \{n \in A : n \text{ is } 2l\text{-th element of } A \text{ for some } l\}$$
$$A'' = \{n \in A : n \text{ is } (2l+1)\text{-th element of } A \text{ for some } l\}$$

For $z \in 2^\omega$, we put

$$w_z(m) = \begin{cases} x(m) & (\text{if } m \in \bigcup_{n \in A'} I_n) \\ z(l) & (\text{if } m \text{ is } l\text{-th element of } \bigcup_{n \in A''} \{\min I_n\}) \\ 1 - y(m) & \text{otherwise} \end{cases}$$

Since $(\forall n \in A')(w_z \upharpoonright I_n = x \upharpoonright I_n)$ holds, we have $w_z \in \mathrm{Match}(x, \mathbb{I})$.

We now prove that $w_z \notin \mathrm{Match}(y, \mathbb{J})$. In order to prove it, let $k \in \omega$.

Suppose that there is an $n \in \omega$ such that $J_k \subseteq I_n$. If $n \in A'$ then we have $w_z \upharpoonright J_k = x \upharpoonright J_k \neq y \upharpoonright J_k$ by $(*)$. If $n \notin A'$, then we have either $n \in A''$ or $n \in \omega \setminus A$. In the former case, $w_z(m) \neq y(m)$ for $m \in J_k \setminus \{\min I_n\}$. Here we used $|J_k| \geq 2$. In the latter case, we have $w_z(m) = 1 - y(m) \neq y(m)$ for every $m \in J_k$.

Suppose that for every $n \in \omega$ we have $J_k \not\subseteq I_n$. Then J_k touches at least 2 intervals in $\mathbb{I}$. At least one of these intervals I_n satisfies $n \notin A$ by $(**)$. Fix such an n. For $m \in J_k \cap I_n$, we have $w_z(m) = 1 - y(m) \neq y(m)$. So we have proved $(\forall k)(w_z \upharpoonright J_k \neq y \upharpoonright J_k)$. Thus, we have $w_z \notin \mathrm{Match}(y, \mathbb{J})$.

Since w_z $(z \in 2^\omega)$ are distinct reals, we are done. $\square$

Lemma 6.7.4. If $\mathcal{F} \subseteq \mathcal{M}$ is a family of size less than $\mathrm{cof}(\mathcal{M})$, then there is a $C \in \mathcal{M}$ such that for all $A \in \mathcal{F}$ we have $|C \smallsetminus A| = \mathfrak{c}$.

Proof. For $A \in \mathcal{F}$, take $x_A \in 2^\omega$ and $\mathbb{I}_A \in \mathsf{IP}$ such that $A \cap \mathrm{Match}(x_A, \mathbb{I}_A) = \varnothing$. Since each $\mathrm{Match}(x_A, \mathbb{I}_A)^{\mathrm{c}}$ is meager set, by the definition of $\mathrm{cof}(\mathcal{M})$, we can take $B \in \mathcal{M}$ such that $B \smallsetminus \mathrm{Match}(x_A, \mathbb{I}_A)^{\mathrm{c}} \neq \varnothing$. Take $y \in 2^\omega$ and $\mathbb{J} \in \mathsf{IP}$ such that $B \cap \mathrm{Match}(y, \mathbb{J}) = \varnothing$. We can assume that $|J_k| \geq 2$ for every $k \in \omega$. Then we have $\mathrm{Match}(y, \mathbb{J})^{\mathrm{c}} \smallsetminus \mathrm{Match}(x_A, \mathbb{I}_A)^{\mathrm{c}} \neq \varnothing$. That is, we have $\mathrm{Match}(x_A, \mathbb{I}_A) \smallsetminus \mathrm{Match}(y, \mathbb{J}) \neq \varnothing$. So by Lemma 6.7.3, $\mathrm{Match}(x_A, \mathbb{I}_A) \smallsetminus \mathrm{Match}(y, \mathbb{J})$ has size $\mathfrak{c}$. Now put $C = \mathrm{Match}(y, \mathbb{J})^{\mathrm{c}}$. Then C is meager and for all $A \in \mathcal{F}$, we have $|C \smallsetminus A| \geq |\mathrm{Match}(x_A, \mathbb{I}_A) \smallsetminus \mathrm{Match}(y, \mathbb{J})| \geq \mathfrak{c}$. So C witnesses the lemma. $\square$

Theorem 6.7.5. $\mathfrak{cp}(\mathcal{M} \smallsetminus \{\varnothing\}) = \mathrm{cof}(\mathcal{M})$.

Proof. This theorem can be shown by the same proof as Theorem 6.6.3 using Lemma 6.7.4 instead of Lemma 6.6.2. $\square$

Theorem 6.7.6. $\mathfrak{icp}(\mathcal{M} \smallsetminus \{\varnothing\}) = \mathrm{add}(\mathcal{M})$.

Proof. This can be shown by the same argument as Theorem 6.6.4. $\square$

Proposition 6.7.7. $\mathfrak{mc}(\mathcal{M}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{mac}(\mathcal{M} \smallsetminus \{\varnothing\}) = \mathfrak{c}$ hold.

Proof. This proposition can be shown by the same argument as Propositions 6.6.5 and 6.6.6. $\square$

6.8 The cardinal invariants of Turing degrees

In this section, we deal with the Turing degrees. Let $\mathcal{D}^+$ denote the poset of all incomputable Turing degrees.

The following fact is well-known.

Fact 6.8.1. $\mathfrak{mac}(\mathcal{D}^+) = \mathfrak{c}$ and $\mathfrak{mc}(\mathcal{D}^+) = \aleph_1$.

Proof. Since $\mathcal{D}^+$ is σ-upward directed, we have that $\mathfrak{mc}(\mathcal{D}^+)$ is uncountable. Moreover, since each downward cone of $\mathcal{D}^+$ is countable, we have $\mathfrak{mc}(\mathcal{D}^+) = \aleph_1$.

Since there are $\mathfrak{c}$ many minimal elements in $\mathcal{D}^+$, we have $\mathfrak{mac}(\mathcal{D}^+) \leq \mathfrak{c}$. Suppose that there is a maximal antichain A of size less than $\mathfrak{c}$ of $\mathcal{D}^+$. Then $A{\downarrow} = \{x \in \mathcal{D}^+ : (\exists y \in A)(x \leq_{\mathrm{T}} y)\}$ has also size less than $\mathfrak{c}$. Thus, we can take a minimal element that does not belong to $A{\downarrow}$. This contradicts maximality of A. $\square$

Using the above fact, we prove the following proposition.

Proposition 6.8.2. $\mathfrak{cp}(\mathcal{D}^+) = \mathfrak{c}$ and $\mathfrak{icp}(\mathcal{D}^+) = \aleph_1$.

Proof. To show $\mathfrak{cp}(\mathcal{D}^+) = \mathfrak{c}$, we fix a comparable family $\mathcal{A} = (A_\alpha : \alpha < \kappa)$. Put $\mathcal{A}' = \{A : A \leq_{\mathrm{T}} A_\alpha \text{ for some } \alpha\}$. Since every downward cone in $\mathcal{D}$ is countable, we have $|\mathcal{A}'| = \kappa$. Fix $B \subseteq \omega$ arbitrarily. Then we can find $\alpha < \kappa$ such that $A_\alpha \leq_{\mathrm{T}} B$ or $B \leq_{\mathrm{T}} A_\alpha$. In either case, we have $(\exists A \in \mathcal{A}')(A \leq_{\mathrm{T}} B)$. So $\mathcal{A}'$ satisfies $(\forall B)(\exists A \in \mathcal{A}')(A \leq_{\mathrm{T}} B)$. So $\mathcal{A}'$ is a coinitial family. But in the poset of Turing degrees, there are continuum many minimal elements. So we have $\mathfrak{cp}(\mathcal{D}^+) \geq \mathfrak{c}$.

Since the poset of Turing degrees is σ-upward directed, we have $\mathfrak{icp}(\mathcal{D}^+) \geq \mathfrak{b}(\mathcal{D}^+) \geq \aleph_1$.

By the previous fact, we have $\mathfrak{icp}(\mathcal{D}^+) \leq \mathfrak{mc}(\mathcal{D}^+) \leq \aleph_1$. ☐

6.9 The cardinal invariants of the Rudin–Keisler ordering

In this section, we will focus on the Rudin–Keisler ordering on the set of nonprincipal ultrafilters on ω.

For the definition and basic properties of Rudin–Keisler ordering, see [Hal12].

Proposition 6.9.1. $\mathfrak{d}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) = 2^{\mathfrak{c}}$.

Proof. Take a dominating family D of $(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}})$. Then we have $\bigcup_{p \in D} p{\downarrow} = \beta\omega \smallsetminus \omega$, where $p{\downarrow}$ is the downward cone below p, whose size is $\leq \mathfrak{c}$. So we have $2^{\mathfrak{c}} \leq \mathfrak{c} \cdot |D|$. Therefore we have $|D| = 2^{\mathfrak{c}}$. ☐

The next lemma is well-known.

Lemma 6.9.2. $\mathfrak{b}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) \geq \mathfrak{c}^+$.

Proof. Let $(p_\alpha : \alpha < \mathfrak{c})$ be a sequence of elements in $\beta\omega \smallsetminus \omega$. We have to show that there is an upper bound of these p_α's. Take an independent family $I = \{f_\alpha : \alpha < \mathfrak{c}\}$ of functions from ω into ω of size $\mathfrak{c}$. By independence, the set

$$\{f_\alpha^{-1}(A) : \alpha < \mathfrak{c}, A \in p_\alpha\}$$

has the strong finite intersection property. So there is an ultrafilter q that extends this set. This q is above all p_α's. ☐

$\mathfrak{b}(\beta\omega \smallsetminus \omega, \geq_{\mathrm{RK}})$ depends on models of set theory. If Near Coherence of Filters (NCF) holds, then $\mathfrak{b}(\beta\omega \smallsetminus \omega, \geq_{\mathrm{RK}}) > 2$, but otherwise $\mathfrak{b}(\beta\omega \smallsetminus \omega, \geq_{\mathrm{RK}}) = 2$.

Proposition 6.9.3. Assume there exist $2^{\mathfrak{c}}$ many Ramsey ultrafilters. Then we have $\mathfrak{cp}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) = 2^{\mathfrak{c}}$.

Proof. Take a comparable family $C \subseteq \beta\omega \smallsetminus \omega$ of size less than $2^{\mathfrak{c}}$. Set $C' = \{p \in \beta\omega \smallsetminus \omega : (\exists q \in C)(p \leq_{\mathrm{RK}} q)\}$. Then C' is a coinitial family. But Ramsey ultrafilters are minimal in $\beta\omega \smallsetminus \omega$. So C' must contain all Ramsey ultrafilters. But the size of C' is less than $2^{\mathfrak{c}}$ because every downward cone has size $\leq \mathfrak{c}$. This contradicts our assumption. ☐

Proposition 6.9.4. In the Miller model over a model of GCH, we have $\mathfrak{d}(\beta\omega \smallsetminus \omega, \geq_{\mathrm{RK}}) \leq \mathfrak{c}$. In particular, $\mathfrak{cp}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) \leq \mathfrak{c}$.

Proof. Note that in the model, NCF holds and there are exactly $\mathfrak{c}$ many P-points. So the set of all P-points is a dominating family of size $\mathfrak{c}$ of the poset $(\beta\omega \smallsetminus \omega, \geq_{\mathrm{RK}})$.

To show this, take an arbitrary ultrafilter p. And take a P-point q. By NCF, there is $r \leq_{\mathrm{RK}} p, q$. Since the property of being a P-point is downward closed, r is also a P-point. So there is a P-point which is below p. ☐

Proposition 6.9.5. $\mathfrak{mc}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) = \mathfrak{b}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) = \mathfrak{c}^+$.

Proof. Take a maximal chain C of $\beta\omega\smallsetminus\omega$. The size of C is less than or equal to $\mathfrak{c}^+$ since each downward cone has size $\leq \mathfrak{c}$. Therefore we have $\mathfrak{mc}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) \leq \mathfrak{c}^+$.

So combining this fact and Lemma 6.9.2, we have

$$\mathfrak{c}^+ \leq \mathfrak{b}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) \leq \mathfrak{mc}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) \leq \mathfrak{c}^+.$$ □

Proposition 6.9.6. If $\mathfrak{icp}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}})$ is defined, then $\mathfrak{icp}(\beta\omega \smallsetminus \omega, \leq_{\mathrm{RK}}) = \mathfrak{c}^+$.

Proof. This follows from Proposition 6.9.5. □

It is a longstanding problem whether it can be proved in ZFC that for every $p \in \beta\omega \smallsetminus \omega$ there is $q \in \beta\omega \smallsetminus \omega$ such that p and q are incomparable. In other words, we don't know whether ZFC proves $\mathfrak{cp}(\beta\omega \smallsetminus \omega) > 1$.

6.10 The cardinal invariants of ideals on ω

In this section, we consider the comparability numbers and incomparability numbers of the ideals on ω. In this section, $A \subseteq^* B$ means $A \smallsetminus B$ is finite for $A, B \subseteq \omega$.

For an ideal $\mathcal{I}$ on ω, recall that the additivity of $\mathcal{I}$, $\mathrm{add}^*(\mathcal{I})$ is defined to be the minimal cardinality of $\mathcal{A} \subseteq \mathcal{I}$ such that for every $B \in \mathcal{I}$ there is $A \in \mathcal{A}$ such that $A \not\subseteq^* B$.

Proposition 6.10.1. Let $\mathcal{I}$ be an ideal on ω that satisfies $\mathsf{fin} \subseteq \mathcal{I}$. Then we have $\mathfrak{icp}(\mathcal{I} \smallsetminus \mathsf{fin}, \subseteq^*) = \mathrm{add}^*(\mathcal{I})$.

Proof. Let $\kappa = \mathrm{add}^*(\mathcal{I})$ and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence of infinite $\mathcal{I}$-small sets such that

$$\neg(\exists C \in \mathcal{I})(\forall \alpha < \kappa)(A_\alpha \subseteq^* C).$$

We construct a sequence $\langle B_i : i < \kappa\rangle$ of infinite $\mathcal{I}$-small sets such that

$$B_i \cap B_{i+1} = \varnothing \text{ for every } i < \kappa \text{ and} \tag{$*$}$$

$$\neg(\exists C \in \mathcal{I})(\forall i < \kappa)(B_i \subseteq^* C). \tag{$**$}$$

We claim that we can take such a sequence. We will construct not only $\langle B_i : i < \kappa\rangle$ but also $\langle \alpha_i : i < \kappa\rangle$. Assume we have constructed B_j and α_j for $j < i$.

If $i = 0$, then put $\alpha_0 = 0$ and $B_0 = A_0$. If i is limit, then put $\alpha_i = \sup_{j<i} \alpha_j$ and $B_i = A_{\alpha_i}$.

Suppose i is a successor ordinal. Find the minimum index $\beta > \alpha_{i-1}$ such that $\neg(A_\beta \subseteq^* A_{\alpha_{i-1}})$ holds. We can take such β, otherwise $\{A_\gamma : \gamma \leq \alpha_{i-1}\}$ is a family in $(\mathcal{I}, \subseteq^*)$ which contradicts $\alpha_{i-1} < \kappa = \mathrm{add}^*(\mathcal{I})$. And we put $\alpha_i = \beta$ and $B_i = A_\beta \smallsetminus A_{\alpha_{i-1}}$.

Then $(*)$ is easily implied from the construction. We have to show $(**)$. Suppose that $(\exists C \in \mathcal{I})(\forall \alpha < \kappa)(B_\alpha \subseteq^* C)$ holds. Take $\alpha < \kappa$ arbitrarily. Take the minimum $i < \kappa$ such that $\alpha < \alpha_i$. This i must be a successor ordinal. Write i as $i = j + n$ where j is a limit ordinal and $n \geq 1$ is a natural number. By the construction, we have $A_\alpha \subseteq^* A_{\alpha_{i-1}}$.

Then we have

$$A_\alpha \subseteq^* A_{\alpha_{i-1}} \subseteq B_j \cup B_{j+1} \cup \cdots \cup B_{j+n} \subseteq^* C.$$

Since α was chosen arbitrarily, this contradicts the choice of the sequence $\langle A_\alpha : \alpha < \kappa \rangle$.

We claim that $\{B_i : i < \kappa\}$ is an incomparable family.

Take an element $C \in \mathcal{I} \smallsetminus \mathsf{fin}$. Then by $(**)$, we can find $i < \kappa$ such that $\neg(B_i \subseteq^* C)$. For this i, if we also have $\neg(C \subseteq^* B_i)$, then we are done. If $C \subseteq^* B_i$, then C and B_{i+1} are almost disjoint, in particular, they are incomparable. $\square$

It is natural to conjecture that $\mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin}, \subseteq^*) = \mathrm{cof}^*(\mathcal{I})$. In the following proposition, we prove it partially.

Proposition 6.10.2. Let $\mathcal{I}$ be a feeble ideal on ω that satisfies $\mathsf{fin} \subseteq \mathcal{I}$. And also assume $\mathrm{cof}^*(\mathcal{I} \upharpoonright A) = \mathrm{cof}^*(\mathcal{I})$ for every $A \in \mathcal{I}^+$. Then we have $\mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin}, \subseteq^*) = \mathrm{cof}^*(\mathcal{I})$.

Lemma 6.10.3. Let $\mathcal{I}$ be a feeble ideal on ω. Then there is an almost disjoint family of size $\mathfrak{c}$ of $\mathcal{I}$-positive sets.

Proof. By Talagrand's theorem, we can take an interval partition $\langle I_n : n \in \omega \rangle$ such that for every $A \in \mathcal{I}$ and for all but finitely many n, we have $I_n \smallsetminus A \neq \varnothing$. Take an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$ of elements in $[\omega]^\omega$. Then the family

$$\mathcal{A}' = \{\bigcup_{n \in A} I_n : A \in \mathcal{A}\}$$

is as desired. $\square$

Proof of Proposition 6.10.2. If $\mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin}) = \mathfrak{c}$ holds, then the conclusion obviously holds. Therefore, we assume $\mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin}) < \mathfrak{c}$. Take a comparable family $C \subseteq \mathcal{I} \smallsetminus \mathsf{fin}$ of size $\mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin})$. By the previous lemma, we can take an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$ of $\mathcal{I}$-positive sets. Since $|C| < |\mathcal{A}|$, we can take $A \in \mathcal{A}$ such that $Y \not\subseteq^* A$ for every $Y \in C$.

We claim that for every $Z \in \mathcal{I} \upharpoonright A$ with $Z \cap A \notin \mathsf{fin}$, there is $Y \in C$ such that $Z \subseteq^* Y \cup A^{\mathrm{c}}$. To prove this claim, fix $Z \in \mathcal{I} \upharpoonright A$ with $Z \cap A \notin \mathsf{fin}$. Then $Z \cap A \in \mathcal{I} \smallsetminus \mathsf{fin}$. Since C is a comparable family, we can take $Y \in C$ such that either $Z \cap A \subseteq^* Y$ or $Y \subseteq^* Z \cap A$ holds. But the latter case must not happen. Thus the former case must happen and $Z \subseteq^* Y \cup A^{\mathrm{c}}$ holds.

By the above claim, $C' := \{Y \cup A^{\mathrm{c}} : Y \in C\}$ is cofinal in $\mathcal{I} \upharpoonright A$. Therefore, we have $\mathrm{cof}^*(\mathcal{I} \upharpoonright A) \leq |C'| \leq |C| = \mathfrak{cp}(\mathcal{I} \smallsetminus \mathsf{fin})$. By the assumption, $\mathrm{cof}^*(\mathcal{I} \upharpoonright A) = \mathrm{cof}^*(\mathcal{I})$ holds and we have the conclusion. $\square$

Corollary 6.10.4. $\mathfrak{cp}(\mathcal{I}_{1/n} \smallsetminus \mathsf{fin}) = \mathrm{cof}^*(\mathcal{I}_{1/n}) = \mathrm{cof}(\mathcal{N})$.

6.11 Weakly ω_1-dense ideals on ω_1

In Section 6.3, we defined $\mathbf{wd}(B)$ for a Boolean algebra B and showed $\mathbf{wd}(B) = \mathfrak{cp}(B \smallsetminus \{0,1\})$ for an atomless Boolean algebra B.

An ideal $\mathcal{I}$ on ω_1 is said to be ω_1 dense if the density of the Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{I}$ is ω_1. Let us define that an ideal $\mathcal{I}$ on ω_1 is weakly ω_1-dense when $\mathbf{wd}(\mathcal{P}(\omega_1)/\mathcal{I}) = \omega_1$ holds.

It is known that the consistency strength of the existence of an ω_1-dense ideal on ω_1 is ω many Woodin cardinals. So it is natural to ask what is the consistency strength of the existence of a weakly ω_1-dense ideal on ω_1. In this section, we answer this question.

Fact 6.11.1 ([BHM73, Theorem 3.1])**.** Let $\mathcal{I}$ be a normal ideal on ω_1. Suppose that $\mathcal{I} \restriction A$ is not ω_1 dense for every $A \in \mathcal{I}^+$. Then for every sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of $\mathcal{I}$-positive sets, there is a pairwise disjoint sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of $\mathcal{I}$-positive sets such that $A_\alpha \subseteq S_\alpha$ for every $\alpha < \omega_1$.

Theorem 6.11.2. Let $\mathcal{I}$ be a normal, weakly ω_1-dense ideal on ω_1. Then $\mathcal{I} \restriction A$ is ω_1-dense for some $A \in \mathcal{I}^+$.

Proof. Suppose that $\mathcal{I} \restriction A$ is not ω_1 dense for every $A \in \mathcal{I}^+$. Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a sequence of $\mathcal{I}$-positive sets. Let us show that this family is not a weakly dense set. So we shall find $B \in \mathcal{I}^+$ such that $S_\alpha \not\subseteq_I B$ and $S_\alpha \not\subseteq_{\mathcal{I}} \omega_1 \setminus B$ for every $\alpha < \omega_1$.

By Fact 6.11.1, we can find a pairwise disjoint sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of $\mathcal{I}$-positive sets such that $A_\alpha \subseteq S_\alpha$ for every $\alpha < \omega_1$. Then we split each A_α into two positive sets B_α, C_α. This can be done using the fact that there is no σ-complete ultrafilter on ω_1. Let B be the union of B_α's. This B is as required. $\square$

Corollary 6.11.3. The consistency strength of the existence of a normal, weakly ω_1-dense ideal on ω_1 is also ω many Woodin cardinals. $\square$

6.12 Open problems

The following questions remain.

Question 6.12.1. (1) What are the values of $\mathfrak{cp}((\mathcal{N} \cap \mathsf{Borel}) \setminus \{\varnothing\})$ and $\mathfrak{cp}((\mathcal{M} \cap \mathsf{Borel}) \setminus \{\varnothing\})$?

(2) Can we prove $\mathfrak{cp}(\mathcal{I} \setminus \mathsf{fin}, \subseteq^*) = \mathrm{cof}^*(\mathcal{I})$ for every ideal on ω? In particular, can we prove this inequality by Tukey reducibility?

(3) What are the values of $\mathfrak{mac}((\mathsf{Borel}(2^\omega)/\mathcal{M})^-)$ and $\mathfrak{mac}((\mathsf{Borel}(2^\omega)/\mathcal{N})^-)$?

(4) In Miller model, what are the values of $\mathfrak{cp}(\beta\omega \setminus \omega, \leq_{\mathrm{RK}})$ and $\mathfrak{mac}(\beta\omega \setminus \omega, \leq_{\mathrm{RK}})$?

(5) Can we prove theorems in Section 6.6 and 6.7 using Tukey reducibility?

Chapter 7

Game-theoretic variants of cardinal invariants

Contents in this chapter is joint work with Jorge Antonio Cruz Chapital, Yusuke Hayashi and Takashi Yamazoe.

The study of cardinal invariants of the continuum is important in set theory of reals. On the other hand, the study of infinite games is also an important topic in set theory. We study variants of cardinal invariants using infinite games. The invariants we treat are the splitting number $\mathfrak{s}$, the reaping number $\mathfrak{r}$, the bounding number $\mathfrak{b}$, the dominating number $\mathfrak{d}$, and the additivity number of the null ideal $\mathrm{add}(\mathcal{N})$.

Depending on the definition of each cardinal invariant, there are normal versions of games and *-versions of games, and we consider 10 games in total.

In the normal version, Player II must in each turn say 0 or 1. Player II wins if there is a real in the prescribed family and the values of this real at the points where Player II chose 1 have the relation to the natural number that Player I said. In contrast, in the *-version, Player II must in each turn choose a natural number. Player II wins if the real consisting of a play of Player II is in the prescribed family and this real has the given relation to the real consisting of Player I's moves.

For each game, two cardinal invariants are defined: the minimum size of a family such that Player II has a winning strategy and the minimum size of a family such that Player I has no winning strategy.

Figure 7.1 summarizes our results.

Game-theoretic considerations of cardinal invariants can be found in [Kad00], [BHT19], and [Sch96] but our approach differs from these.

$\mathbb{0}$ is the set of all eventually zero sequences and $\mathbb{1}$ is that of eventually one sequences.

7.1 Bounding games

In this section, we consider games related to unbounded families.

Fix a set $\mathcal{A} \subseteq \omega^\omega$. We call the following game the *bounding game* with respect to $\mathcal{A}$:

$$\begin{array}{c|cccccc} \text{Player I} & n_0 & & n_1 & & \dots & \\ \hline \text{Player II} & & i_0 & & i_1 & & \dots \end{array}$$

game	$\mathfrak{x}^{\mathrm{I}}_{\mathrm{game}}$	$\mathfrak{x}^{\mathrm{II}}_{\mathrm{game}}$
bounding	$\mathfrak{b}$	$\mathfrak{d}$
bounding*	$\mathfrak{b}$	$\mathfrak{c}$
dominating	$\mathfrak{d}$	$\mathfrak{d}$
dominating*	$\mathfrak{d}$	$\mathfrak{c}$
splitting	$\mathfrak{s}_\sigma$	$\mathfrak{c}$
splitting*	$\mathfrak{s}_\sigma \le ? \le \min\{\mathrm{non}(\mathcal{M}), \mathfrak{d}, \mathrm{non}(\mathcal{N})\}$	$\mathfrak{c}$
reaping	$\max\{\mathfrak{r}, \mathfrak{d}\} \le ? \le \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$	$\mathfrak{c}$
reaping*	∞	∞
anti-localizing	$\mathrm{add}(\mathcal{N})$	$\mathrm{cov}(\mathcal{M})$
anti-localizing*	$\mathrm{add}(\mathcal{N})$	$\mathfrak{c}$

Figure 7.1: Our results

Here, $\langle n_k : k \in \omega\rangle$ is a sequence of numbers in ω and $\langle i_k : k \in \omega\rangle$ is a sequence of numbers in 2. Player II wins when Player II played 1 infinitely often and there is $g \in \mathcal{A}$ such that

$$\{k \in \omega : i_k = 1\} = \{k \in \omega : n_k < g(k)\}.$$

We call the following game the *bounding* game* with respect to $\mathcal{A}$:

Player I	n_0		n_1		$\dots$	
Player II		m_0		m_1		$\dots$

Here, $\langle n_k : k \in \omega\rangle$ and $\langle m_k : k \in \omega\rangle$ are sequences of numbers in ω. Player II wins when

$$\langle m_k : k \in \omega\rangle \in \mathcal{A} \text{ and } (\exists^\infty k)(n_k < m_k).$$

Definition 7.1.1. We define

$$\begin{aligned}
\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}} &= \min\{|\mathcal{A}| : \text{Player I has no winning strategy} \\
&\qquad\qquad \text{for the bounding game with respect to } \mathcal{A}\}, \\
\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}} &= \min\{|\mathcal{A}| : \text{Player II has a winning strategy} \\
&\qquad\qquad \text{for the bounding game with respect to } \mathcal{A}\}, \\
\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*} &= \min\{|\mathcal{A}| : \text{Player I has no winning strategy} \\
&\qquad\qquad \text{for the bounding* game with respect to } \mathcal{A}\}, \text{ and} \\
\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}^*} &= \min\{|\mathcal{A}| : \text{Player II has a winning strategy} \\
&\qquad\qquad \text{for the bounding* game with respect to } \mathcal{A}\}.
\end{aligned}$$

Since the star version is harder for Player II than the non-star version, we have the following inequality.

$$\begin{array}{ccc} \mathfrak{b}^{\mathrm{II}}_{\mathrm{game}} & \leq & \mathfrak{b}^{\mathrm{II}}_{\mathrm{game}^*} \\ \vee\!\mathrm{I} & & \vee\!\mathrm{I} \\ \mathfrak{b}^{\mathrm{I}}_{\mathrm{game}} & \leq & \mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*} \end{array}$$

Theorem 7.1.2. $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}} = \mathfrak{b}$ holds.

Proof. That $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}} \geq \mathfrak{b}$ is easy. We show $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}} \leq \mathfrak{b}$. Take an unbounded family $\mathcal{A} \subseteq \omega^\omega$. Take Player I's strategy $\sigma\colon 2^{<\omega} \to \omega$. We want to show that σ is not a winning strategy for the bounding game with respect to $\mathcal{A}$.

Since $\mathbb{0}$ is a countable set, we can get $f \in \omega^\omega$ that dominates $\langle \sigma(\bar{i} \restriction k) : k \in \omega \rangle$ for every $\bar{i} \in \mathbb{0}$. Since $\mathcal{A}$ is an unbounded family, we can take $g \in \mathcal{A}$ such that f doesn't dominate g. We now put $\bar{i} \in 2^\omega$ by

$$i_k = \begin{cases} 1 & (\text{if } \sigma(\bar{i} \restriction k) < g(k)) \\ 0 & (\text{otherwise}) \end{cases}$$

If $\bar{i} \in \mathbb{0}$, then $\langle \sigma(\bar{i} \restriction k) : k \in \omega \rangle$ does not dominate g by the choice of g. But this fact and the choice of $\bar{i}$ imply $\bar{i} \notin \mathbb{0}$. It's a contradiction. So $\bar{i} \notin \mathbb{0}$. Therefore $\bar{i}$ is a play of Player II that wins against Player I's strategy σ. □

Theorem 7.1.3. $\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}} = \mathfrak{d}$ holds.

Proof. We first prove $\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}} \leq \mathfrak{d}$. Take a dominating family $\mathcal{A} \subseteq \omega^\omega$ of $(\omega^\omega, \leq)$ (the total domination order). Then the strategy that says 1 always is a winning strategy for Player II.

We next prove $\mathfrak{d} \leq \mathfrak{b}^{\mathrm{II}}_{\mathrm{game}}$. Fix $\mathcal{A} \subseteq \omega^\omega$ with a winning strategy of Player II for the bounding game with respect to $\mathcal{A}$. Consider the game tree T decided by the winning strategy. So every node in T of even length has full successor nodes and every node in T of odd length has the only successor node determined by the strategy. We first consider the next case:

- (Case 1) There is a $\sigma \in T$ of even length such that for every even number $r \geq |\sigma|$, there is $i \in 2$ such that for all but finitely many m, for every $\tau \in T$ extending σ, we have $[\tau(r) = m \implies \tau(r+1) = i]$.

Fix a witness σ and $\langle i_r : r \geq |\sigma| \text{ even} \rangle$ for Case 1.

Then we have $(\exists^\infty r)(i_r = 1)$. Otherwise, we have $(\forall^\infty r)(i_r = 0)$. Then considering an appropriate play of Player I, Player II says 0 eventually along the winning strategy. This is a contradiction to the rule of the game.

Consider the increasing enumeration $\{r_n : n \in \omega\}$ of $\{r \in \omega : i_r = 1\}$. For each $n \in \omega$, we have $m_n \in \omega$ satisfying for every $\tau \in T$ extending σ, we have $[\tau(r_n) \geq m_n \implies \tau(r_n + 1) = 1]$. Fix $f \in \omega^\omega$. Consider the play of Player I that plays $\max\{m_n, f(n)\}$ at stage $r_n/2$. Since Player II wins, there is $g \in \mathcal{A}$ such that

$$\max\{m_n, f(n)\} \leq g(r_n/2).$$

So $\mathcal{A}' = \{\langle g(r_n/2) : n \in \omega \rangle : g \in \mathcal{A}\}$ is a dominating family. We have $|\mathcal{A}| \geq \mathfrak{d}$.

We next consider the next case:

- (Case 2) For every $\sigma \in T$ of even length, there is an even number $r \geq |\sigma|$ such that for every $i \in 2$, there exist infinitely many m and there is $\tau \in T$ extending σ such that $[\tau(r) = m \wedge \tau(r+1) = i]$.

In this case, we can construct a perfect subtree of T and each distinct path of this subtree gives a distinct element of $\mathcal{A}$.

In detail, we construct r_s, σ_s and $m_0^s < m_1^s$ for $s \in 2^{<\omega}$ such that $\sigma_{s^\frown i}(r_s) = m_i^s$, $\sigma_{s^\frown i}(r_s+1) = i$ for every $i < 2$. For each $f \in 2^\omega$, put $\sigma_f = \bigcup_{n\in\omega} \sigma_{f\restriction n}$. Since II wins, we can take $g_f \in \mathcal{A}$ that witnesses σ_f is a winning play. Take distinct f and f' in 2^ω. Let $\Delta = \min\{n : f(n) \neq f'(n)\}$ and $s = f \restriction \Delta = f' \restriction \Delta$. We may assume that $f(\Delta) = 0$ and $f'(\Delta) = 1$. We have $\sigma_f(r_s) = m_0^s, \sigma_f(r_s + 1) = 0, \sigma_{f'}(r_s) = m_1^s$ and $\sigma_{f'}(r_s + 1) = 1$. Then by the rule of the game, we have

$$g_f(r_s/2) \leq m_0^s < m_1^s < g_{f'}(r_s/2).$$

So we have $g_f \neq g_{f'}$. Therefore, we have $|\mathcal{A}| = \mathfrak{c}$ in this case.

In either case, we have $|\mathcal{A}| \geq \mathfrak{d}$, so we have shown $\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}} \geq \mathfrak{d}$. $\square$

Using terminology in [Bla10, Section 10], $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*}$ is equal to the global, adaptive, finite prediction version of the evasion number. Moreover, in the article it was shown that this invariant is equal to $\mathfrak{b}$. So we have $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*} = \mathfrak{b}$. But for the sake of completeness, we include the proof.

Theorem 7.1.4. $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*} = \mathfrak{b}$ holds.

Proof. It is clear that $\mathfrak{b} \leq \mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*}$. We show $\mathfrak{b}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathfrak{b}$.

Take an unbounded family $\mathcal{A}$ of ω^ω. Take an arbitrary strategy $\sigma \colon \omega^{<\omega} \to \omega$ of Player I. We have to show that σ is not a winning strategy for the bounding* game with respect to $\mathcal{A}$.

Fix an enumeration $\langle s_i : i \in \omega\rangle$ of $\omega^{<\omega}$ that satisfies $|s_i| \leq i$ for every i. For each $s \in \omega^{<\omega}$ and $n \in \omega \setminus |s|$, we put

$$\sigma_s(n) = \max\{x(n) : s \subseteq x \in \omega^\omega \text{ and } (\forall k \geq |s|)(x(k) \leq \sigma(x \restriction k)\}.$$

It can be easily checked that $\sigma_s(n)$ is in ω. We define f by

$$f(n) = \max(\{\sigma_{s_i}(n) : i < n\} \cup \{0\}).$$

Take $g \in \mathcal{A}$ that is not dominated by f. Consider the play in which Player I obeys the strategy σ and Player II plays g. Suppose that Player I wins. Then there is $n_0 \in \omega$ such that $(\forall n \geq n_0)(g(n) \leq \sigma(g \restriction n))$. Take $m_0 \in \omega$ such that $s_{m_0} = g \restriction n_0$. Then we have for every $m > m_0$:

$$g(m) \leq \sigma_{s_{m_0}}(m) \leq f(m).$$

This means that f dominates g, which is a contradiction. $\square$

Theorem 7.1.5. $\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$ holds.

Proof. Fix $\mathcal{A} \subseteq \omega^\omega$ such that Player II has a winning strategy τ for the bounding* game with respect to $\mathcal{A}$. We shall show that $\mathcal{A}$ is of size $\mathfrak{c}$. Consider the game tree $T \subseteq \omega^{<\omega}$ that the strategy determines.

First, assume the following.

- (Case 1) There is a $\sigma \in T$ such that for every odd $k \geq |\sigma|$, there is an $m_k < \omega$ such that for every $\tau \in T$ extending σ with $|\tau| > k$, we have $\tau(k) = m_k$.

Fix the witness $\sigma, \langle m_k : k \geq |\sigma|\rangle$ for Case 1.

Consider the next play.

Player I	$\sigma(0)$		$\dots$	$\sigma(\lvert\sigma\rvert-2)$		$m_{\lvert\sigma\rvert}$		$m_{\lvert\sigma\rvert+2}$		$\dots$
Player II		$\sigma(1)$	$\dots$		$\sigma(\lvert\sigma\rvert-1)$		$m_{\lvert\sigma\rvert}$		$m_{\lvert\sigma\rvert+2}$	

Then the sequence defined by the play of Player II does not dominate that defined by the play of Player I. So Player II loses. This is a contradiction.

So Case 1 is false. Thus we have

- (Case 2) For every $\sigma \in T$, there is an odd number $k \geq |\sigma|$ such that for every $m < \omega$, there is $\tau \in T$ extending σ with $|\tau| > k$ such that $\tau(k) \neq m$.

Note that there are $\tau_0, \tau_1 \supseteq \sigma$ with $|\tau_0|, |\tau_1| > k$ such that $\tau_0(k) \neq \tau_1(k)$ in Case 2.

Now we can construct a subtree of T in the following manner. First we put $\sigma_\varnothing = \varnothing$. Suppose we have $\langle \sigma_s : s \in 2^{\leq l} \rangle$. Then for each $s \in 2^l$, we can take $\sigma_{s^\frown 0}, \sigma_{s^\frown 1} \supseteq \sigma_s$ and $k_s \geq |\sigma_s|$ such that $\sigma_{s^\frown 0}(k_s) \neq \sigma_{s^\frown 1}(k_s)$.

Now for each $f \in 2^\omega$, we put σ_f by $\sigma_f = \bigcup_{n\in\omega} \sigma_{f \upharpoonright n}$.

For each $f \in 2^\omega$, we have $\sigma_f \in [T]$. So Player II wins at the play σ_f. So by the definition of the game, we can take $x_f \in \mathcal{A}$ such that $x_f(k) = \sigma_f(2k+1)$.

We now claim that if f and g are distinct elements of 2^ω, then we have $x_f \neq x_g$. Let $n := \min\{n' : f(n') \neq g(n')\}$. Put $s = f \upharpoonright n = g \upharpoonright n$. We may assume that $f(n) = 0$ and $g(n) = 1$. We have the following:

$$x_f\left(\frac{k_s - 1}{2}\right) = \sigma_f(k_s) = \sigma_{s^\frown 0}(k_s) \neq \sigma_{s^\frown 1}(k_s) = \sigma_g(k_s) = x_g\left(\frac{k_s - 1}{2}\right).$$

So we have $x_f \neq x_g$.

Therefore we have $|\mathcal{A}| \geq |\{x_f : f \in 2^\omega\}| = \mathfrak{c}$. $\square$

7.2 Dominating games

In this section, we consider games related to dominating families.

Fix a set $\mathcal{A} \subseteq \omega^\omega$. We call the following game the *dominating game* with respect to $\mathcal{A}$:

Player I	n_0		n_1		$\dots$	
Player II		i_0		i_1		$\dots$

Here, $\langle n_k : k \in \omega \rangle$ is a sequence of numbers in ω and $\langle i_k : k \in \omega \rangle$ is a sequence of numbers in 2. Player II wins when Player II played 1 eventually and there is $g \in \mathcal{A}$ such that

$$\{k \in \omega : i_k = 1\} = \{k \in \omega : n_k < g(k)\}.$$

We call the following game the *dominating* game* with respect to $\mathcal{A}$:

Player I	n_0		n_1		$\dots$	
Player II		m_0		m_1		$\dots$

Here, $\langle n_k : k \in \omega \rangle$ and $\langle m_k : k \in \omega \rangle$ are sequences of numbers in ω. Player II wins when

$$\langle m_k : k \in \omega \rangle \in \mathcal{A} \text{ and } (\forall^\infty k)(n_k < m_k).$$

We define $\mathfrak{d}^{\mathrm{I}}_{\mathrm{game}}, \mathfrak{d}^{\mathrm{II}}_{\mathrm{game}}, \mathfrak{d}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{d}^{\mathrm{II}}_{\mathrm{game}^*}$ by using dominating games and dominating* games in the same fashion as Definition 7.1.1.

Theorem 7.2.1. $\mathfrak{d}^{\mathrm{I}}_{\mathrm{game}} = \mathfrak{d}^{\mathrm{II}}_{\mathrm{game}} = \mathfrak{d}^{\mathrm{I}}_{\mathrm{game}^*} = \mathfrak{d}$ and $\mathfrak{d}^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$ hold.

Proof. $\mathfrak{d} \leq \mathfrak{d}^{\mathrm{I}}_{\mathrm{game}}$ is easy. $\mathfrak{d}^{\mathrm{II}}_{\mathrm{game}} \leq \mathfrak{d}$ follows from the observation that for every totally dominating family $\mathcal{A}$, Player II has a winning strategy for the dominating game with respect to $\mathcal{A}$. So we have $\mathfrak{d}^{\mathrm{I}}_{\mathrm{game}} = \mathfrak{d}^{\mathrm{II}}_{\mathrm{game}} = \mathfrak{d}$.

$\mathfrak{d}^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$ follows from $\mathfrak{b}^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$ which was shown in Theorem 7.1.5, since the dominating* game is harder for Player II than the bounding* game.

We know $\mathfrak{d} = \mathfrak{d}^{\mathrm{I}}_{\mathrm{game}} \leq \mathfrak{d}^{\mathrm{I}}_{\mathrm{game}^*}$. So the remaining work is to show $\mathfrak{d}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathfrak{d}$. To show it, let $\pi\colon \omega \to \omega^{<\omega}$ be a bijection. Fix a dominating family $\mathcal{F} \subseteq \omega^\omega$. For $g \in \mathcal{F}$, we define $g' \in \omega^\omega$ so that

$$(\forall n)((g \circ \pi^{-1})(g' \restriction n) < g'(n)).$$

This g' can be constructed by induction on n. Put

$$\mathcal{A} = \{g' : g \in \mathcal{F}\}.$$

Take an arbitrary strategy $\sigma\colon \omega^{<\omega} \to \omega$ of Player I. We have to show that σ is not a winning strategy. Since $\mathcal{F}$ is a dominating family, we can take $g \in \mathcal{F}$ that dominates $\sigma \circ \pi$. Then for all but finitely many m, we have

$$\sigma(g' \restriction n) = \sigma(\pi(\pi^{-1}(g' \restriction n))) \leq g(\pi^{-1}(g' \restriction n)) \leq g'(n).$$

This inequality means if Player II plays g', then Player II wins against Player I who obeys the strategy σ. So we have proved σ is not a winning strategy. $\square$

7.3 Splitting games

In this section, we consider games related to splitting families. Moreover, using such games, we find a new cardinal invariant $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ that differs from previously studied cardinal invariants related to $\mathfrak{s}$.

Fix a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$. We call the following game the *splitting game* with respect to $\mathcal{A}$:

Player I	n_0		n_1		$\dots$	
Player II		i_0		i_1		$\dots$

Here, $n_0 < n_1 < n_2 < \dots < n_k < \dots$ are increasing numbers in ω, i_k $(k \in \omega)$ are elements in $\{0, 1\}$. Player II wins when Player II played each of 0 and 1 infinitely often and there is $A \in \mathcal{A}$ such that

$$\{n_k : k \in \omega\} \cap A = \{n_k : k \in \omega \text{ and } i_k = 1\}. \tag{$*$}$$

Fix a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$. We call the following game the *splitting* game* with respect to $\mathcal{A}$:

Player I	i_0		i_1		$\dots$	
Player II		j_0		j_1		$\dots$

Here, $\langle i_k : k \in \omega\rangle$ and $\langle j_k : k \in \omega\rangle$ are sequences of elements in $\{0, 1\}$. Player II wins when either Player I did not play 1 infinitely often or

$$\{k \in \omega : j_k = 1\} \in \mathcal{A} \text{ and splits } \{k \in \omega : i_k = 1\}.$$

We define $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}}, \mathfrak{s}^{\mathrm{II}}_{\mathrm{game}}, \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{II}}_{\mathrm{game}^*}$ by using splitting games and splitting* games in the same fashion as Definition 7.1.1.

The splitting* game is harder for Player II than the splitting game. More precisely,

Lemma 7.3.1. (1) If Player II has a winning strategy for the splitting* game with respect to $\mathcal{A}$, then Player II has a winning strategy for the splitting game with respect to $\mathcal{A}$.

(2) If Player I has a winning strategy for the splitting game with respect to $\mathcal{A}$, then Player I has a winning strategy for the splitting* game with respect to $\mathcal{A}$.

We omit the proof of this lemma. We can deduce from this lemma that $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{II}}_{\mathrm{game}} \leq \mathfrak{s}^{\mathrm{II}}_{\mathrm{game}^*}$.

Theorem 7.3.2. $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}} = \mathfrak{s}_\sigma$ holds.

Proof. First we prove $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathfrak{s}_\sigma$. Fix a σ-splitting family $\mathcal{A} \subseteq [\omega]^\omega$. We want to show that Player I has no winnng strategy for the splitting game with respect to $\mathcal{A}$. Fix a strategy $\sigma \colon 2^{<\omega} \to \omega$ of Player I. Since $\mathbb{0} \cup \mathbb{1}$ is a countable set and $\mathcal{A}$ is a σ-splitting family, we can take $A \in \mathcal{A}$ such that A splits all $\{\sigma(\bar{i} \restriction k) : k \in \omega\}$ for $\bar{i} \in \mathbb{0} \cup \mathbb{1}$.

We consider the following $\bar{i} \in 2^\omega$:

$$i_k = \begin{cases} 1 & (\text{if } \sigma(\bar{i} \restriction k) \in A) \\ 0 & (\text{otherwise}) \end{cases}.$$

If $\bar{i} \in \mathbb{0} \cup \mathbb{1}$, then A splits $\{\sigma(\bar{i} \restriction k) : k \in \omega\}$ by the choice of A. But by the choice of $\bar{i}$, this means $\bar{i} \notin \mathbb{0} \cup \mathbb{1}$, which is a contradiction. So we have $\bar{i} \notin \mathbb{0} \cup \mathbb{1}$. This observation and the choice $\bar{i}$ imply $\bar{i}$ is a winning play of Player II against the strategy σ. So we have proved Player I has no winning strategy for the splitting game with respect to $\mathcal{A}$.

Next, we prove $\mathfrak{s}_\sigma \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}}$. Fix a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that Player I has no winning strategy for the splitting game with respect to $\mathcal{A}$. We want to show that $\mathcal{A}$ is a σ-splitting family. Take $f \colon \omega \to [\omega]^\omega$. We shall find an $A \in \mathcal{A}$ such that A splits $f(n)$ for every $n \in \omega$. Take $f' \colon \omega \to [\omega]^\omega$ such that $\mathrm{ran}(f) = \mathrm{ran}(f')$ and each element of $\mathrm{ran}(f)$ appears in the range of f' infinitely often. For $m, n \in \omega$, we let $f'(n)(m)$ denote the m-th element of $f'(n)$ in ascending order.

Consider the following strategy σ of Player I. First σ plays $f'(0)(0)$.

From then on, σ will play the elements of $f'(0)$ in turn until Player II changes the value of play. After that, σ plays $f'(1)(k)$ next. Here k is the smallest number such that $f'(1)(k)$ exceeds the natural number that σ has said so far. Continue this process.

The following table is an example:

Player I	$f'(0)(0)$		$f'(0)(1)$		$f'(0)(2)$		$f'(1)(k)$	$\dots$
Player II		0		0		1		$\dots$

Since this σ is not a winning strategy, there is $A \in \mathcal{A}$ and $\bar{i} \in 2^\omega \setminus (\mathbb{0} \cup \mathbb{1})$ such that the equation $(*)$ holds for $n_k = \sigma(\bar{i} \restriction k)$. This implies A splits all elements in $\mathrm{ran}(f)$ by the definition of σ. $\square$

Theorem 7.3.3. $\mathfrak{s}^{\mathrm{II}}_{\mathrm{game}} = \mathfrak{c}$ holds.

Proof. Fix $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that Player II has a winning strategy for the splitting game with respect to $\mathcal{A}$. We shall show that $\mathcal{A}$ is of size $\mathfrak{c}$. Consider the game tree $T \subseteq \omega^{<\omega}$ that the winning strategy determines.

First, assume the following.

- (Case 1) There is an even number $k \in \omega$ and there is a $\sigma \in \omega^k \cap T$ such that for every $m > \sigma(k-2)$ there is $i_m < 2$ such that for every $\tau \in T$ extending σ and every $r \in [|\sigma|, |\tau|)$, $\tau(r) = m$ implies $\tau(r+1) = i_m$.

Fix the witness $k, \sigma, \langle i_m : m > \sigma(k-2)\rangle$ for Case 1. Take an infinite set $A \subseteq [\sigma(k-2), \omega)$ and $i^* < 2$ such that $i_m = i^*$ for every $m \in A$. Enumerate A in ascending order as $A = \{a_i : i \in \omega\}$.

Then considering the Player I's play that says $a_0, a_1, a_2, \ldots$ in turn after σ, Player II that obeys the winning strategy plays i^* eventually. So Player II loses, which is a contradiction.

So Case 1 is false. Thus we have

- (Case 2) For every even number $k \in \omega$ and every $\sigma \in \omega^k \cap T$, there is $m > \sigma(k-2)$ such that for every $i < 2$, there is $\tau \in T$ extending σ and there is $r \in [|\sigma|, |\tau|)$ such that $\tau(r) = m$ and $\tau(r+1) = 1 - i$.

Then we can construct a perfect subtree of T whose distinct paths yield distinct elements of $\mathcal{A}$ by using a method similar to Theorem 7.1.3 and 7.1.5.

Therefore we have $|\mathcal{A}| \geq |\{A_f : f \in 2^\omega\}| = \mathfrak{c}$. □

By the remark below Lemma 7.3.1, we have also $\mathfrak{s}_\sigma \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$.

In the following theorem, we give an upper bound of $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$.

Theorem 7.3.4. $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{M})$ holds.

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be a non-meager set. Set $\mathcal{A}' = \mathcal{A} \cup \mathbb{0} \cup \mathbb{1}$. We prove that Player I has no winning strategy for the splitting* game with respect to $\mathcal{A}'$. Take an arbitrary strategy σ of Player I. Define for each $x \in 2^\omega$, $y_x = \{k \in \omega : \sigma(x \restriction k) = 1\}$.

Consider the following case:

- (Case 1) There is $s \in 2^{<\omega}$ such that $\sigma(s^\frown \langle 0\rangle^m) = 0$ for every m.

In this case, if Player II plays $s^\frown\langle 0, 0, \ldots\rangle$, then Player II wins against Player I that obeys σ, since Player I plays 0 eventually.

Next, consider the following case:

- (Case 2) There is $s \in 2^{<\omega}$ such that $\sigma(s^\frown \langle 1\rangle^m) = 0$ for every m.

In this case, we also have the same conclusion as in Case 1.

So deny Case 1 and 2. Define a set $\mathcal{C}$ as follows:

$$\mathcal{C} = \{x \in 2^\omega : x \text{ splits } y_x\}.$$

We claim that $\mathcal{C}$ is a comeager set of 2^ω.

For $m \in \omega$, consider the following sets:

$$\mathcal{D}_m = \{x \in 2^\omega : (\exists n \geq m)(x(n) = 1 \text{ and } \sigma(x \restriction n) = 1)\},$$
$$\mathcal{E}_m = \{x \in 2^\omega : (\exists n \geq m)(x(n) = 0 \text{ and } \sigma(x \restriction n) = 1)\}.$$

$\mathcal{D}_m$ is an open dense set since we denied Case 2. Similarly, $\mathcal{E}_m$ is an open dense set since we denied Case 1.

So the set

$$\mathcal{C} = \bigcap_m \mathcal{D}_m \cap \bigcap_m \mathcal{E}_m$$

is comeager. Since $\mathcal{A}$ is non-meager, $\mathcal{A} \cap \mathcal{C} \neq \varnothing$. An element in $\mathcal{A} \cap \mathcal{C}$ is a play of Player II that wins against Player I that obeys σ. □

Theorem 7.3.5. $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathfrak{d}$ holds.

Proof. Let $\{I_\xi = \langle I^\xi_n \mid n < \omega\rangle \mid \xi < \mathfrak{d}\}$ be a dominating family with respect to interval partitions. Define $x_\xi = \bigcup_{n<\omega} I^\xi_{2n}$ and set $\mathcal{A} = \{x_\xi \mid \xi < \mathfrak{d}\}$. We prove that Player I has no winning strategy for the splitting* game with respect to $\mathcal{A} \cup \mathbb{0} \cup \mathbb{1}$. Take an arbitrary strategy σ of Player I.

Consider the following case:

- (Case 1) There is $s \in 2^{<\omega}$ such that $\sigma(s ^\frown \langle 0\rangle^m) = 0$ for every $m > 0$.

In this case, if Player II plays $s ^\frown \langle 0, 0, \dots\rangle$, then Player II wins against Player I that obeys σ, since Player I plays 0 eventually.

Next, consider the following case:

- (Case 2) There is $s \in 2^{<\omega}$ such that $\sigma(s ^\frown \langle 1\rangle^m) = 0$ for every $m > 0$.

In this case, we also have the same conclusion as in Case 1.

So we can assume the following:

- (Case 3) For all $s \in 2^{<\omega}$ and $i < 2$, there is $m^i_s > 0$ such that $\sigma(s ^\frown \langle i\rangle^{m^i_s}) = 1$.

Fix these m^i_s's and define a sequence $\langle j_k \mid k < \omega\rangle$ of natural numbers as follows:

$$j_0 = 0$$
$$j_{2k+i} = j_{2k+i-1} + \max\{m^i_s \mid s \in 2^{j_{2k+i-1}}\} \text{ for each } i < 2.$$

Let $J_k = [j_{2k}, j_{2k+2})$. Since $\{\langle I^\xi_n \mid n < \omega\rangle \mid \xi < \mathfrak{d}\}$ is a dominating family, there is I_ξ such that

$$(\exists n_0 < \omega)(\forall n \geq n_0)(\exists k)(J_k \subseteq I^\xi_n).$$

Let $I^\xi_n = [i^\xi_n, i^\xi_{n+1})$. Take an arbitrary $n \geq n_0$ and k such that $J_k \subseteq I^\xi_n$. Note that $j_{2k}, j_{2k+1} \in J_k$. First, we consider the case n is even. Set $m = m^1_{x_\xi \restriction j_{2k}} > 0$. By the construction of j_{2k+1}, we have $j_{2k} + m \leq j_{2k+1} \leq i^\xi_{n+1}$. So it holds that $\sigma(x_\xi \restriction (j_{2k} + m)) = \sigma((x_\xi \restriction j_{2k}) ^\frown \langle 1\rangle^m) = 1$. Thus, we have $x_\xi(j_{2k} + m) = 1$ and $\sigma(x_\xi \restriction (j_{2k} + m)) = 1$.

In the case n is odd, by a similar argument, we have that there is $m > 0$ such that $x_\xi(j_{2k} + m) = 0$ and $\sigma(x \restriction (j_{2k} + m)) = 1$.

Therefore x_ξ is a play of Player II that wins against Player I that obeys σ. □

Theorem 7.3.6. $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{N})$ holds.

To prove this theorem, we prepare some lemmas.

Lemma 7.3.7. Let $I = [i, j)$ be an interval in ω. Let $s \in \{0,1\}^i$, $\sigma\colon \{0,1\}^{<j} \to 2$, and $\varepsilon \in 2$. Set

$$\begin{aligned} B^I_{s,\varepsilon}(\sigma) = \{x \in \{0,1\}^j : & s \subseteq x, (\exists k \in I)(\sigma(x \restriction k) = 1), \text{ and} \\ & (\forall k \in I)(\sigma(x \restriction k) = 1 \to x(k) = \varepsilon)\}. \end{aligned}$$

Then we have

$$\frac{|B^I_{s,\varepsilon}(\sigma)|}{2^{j-i}} \leq \frac{1}{2}.$$

Proof. Induction on $|I|$. If $|I| = 1$ then $|B^I_{s,\varepsilon}(\sigma)| = |\{s^\frown\langle\varepsilon\rangle\}| = 1$. So in this case, the lemma is proven. Suppose $|I| \geq 2$. If $\sigma(s) = 1$, then $|B^I_{s,\varepsilon}(\sigma)| \leq |\{x : s^\frown\langle\varepsilon\rangle \subseteq x\}| = 2^{j-i-1}$. Otherwise, by the induction hypothesis, we have

$$|B^I_{s,\varepsilon}(\sigma)| = |B^{[i+1,j)(\sigma)}_{s^\frown\langle 0\rangle,\varepsilon}(\sigma)| + |B^{[i+1,j)}_{s^\frown\langle 1\rangle,\varepsilon}(\sigma)| \leq 2^{j-(i+1)-1} + 2^{j-(i+1)-1} = 2^{j-i-1}.$$

□

Lemma 7.3.8. Let $a < b < \omega$. Let $\bar{I} = \langle I_n : a \leq n < b\rangle$ be a sequence of consecutive intervals in ω and put $m := \min I_a$ and $M := \max I_{b-1} + 1$. Let $\sigma\colon \{0,1\}^{<M} \to 2$ and $\varepsilon \in 2$. Set

$$\begin{aligned} B^{\bar{I}}_{\varepsilon}(\sigma) = \{x \in \{0,1\}^M : & (\forall n \in [a,b))[(\exists k \in I_n)(\sigma(x \restriction k) = 1), \text{ and} \\ & (\forall k \in I_n)(\sigma(x \restriction k) = 1 \to x(k) = \varepsilon)]\}. \end{aligned}$$

Then we have

$$\frac{|B^{\bar{I}}_{\varepsilon}(\sigma)|}{2^M} \leq \frac{1}{2^{b-a}}.$$

Proof. Use the previous lemma and induct on $b - a$. □

We use the following theorem due to Goldstern, which we introduce in Chapter 3.

Fact 7.3.9 ([Gol93])**.** Let $A \subseteq \mathsf{IP} \times 2^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Suppose that the vertical section $A_{\bar{I}}$ is null for every $\bar{I} \in \mathsf{IP}$ and $A_{\bar{I}} \subseteq A_{\bar{J}}$ for every $\bar{I}, \bar{J} \in \mathsf{IP}$ with $\bar{I} < \bar{J}$. Then $\bigcup_{\bar{I}\in\mathsf{IP}} A_{\bar{I}}$ is null.

Goldstern proved this theorem not with IP, but with ω^ω. But these two versions can easily be shown to be equivalent.

Proof of Theorem 7.3.6. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be a non-null set of size $\mathrm{non}(\mathcal{N})$. We will show that Player I has no winning strategy for the splitting* game with respect to $\mathcal{A}$. Fix a strategy $\sigma\colon 2^{<\omega} \to 2$ of Player I.

For $\bar{I} \in \mathsf{IP}$ and $\varepsilon \in 2$, define

$$C^{\bar{I}}_{\varepsilon} = \bigcup_{a\in\omega} \bigcap_{b>a} B^{\bar{I}\restriction[a,b)}_{\varepsilon}(\sigma).$$

By Lemma 7.3.8, this set $C^{\bar{I}}_{\varepsilon}$ is null.

Moreover when $\bar{I} < \bar{J}$, we have $C^{\bar{I}}_{\varepsilon} \subseteq C^{\bar{J}}_{\varepsilon}$. Also, the set $\{(\bar{I}, x) : x \in C^{\bar{I}}_{\varepsilon}\}$ is clearly a Borel set. Therefore, we can apply Goldstern's theorem to get that $\bigcup_{\bar{I}\in\mathsf{IP}} C^{\bar{I}}_{\varepsilon}$ is null.

Moreover, we can easily observe that

$$\{x \in 2^\omega : \text{the strategy } \sigma \text{ wins the play } x \text{ in splitting* game}\} \subseteq \bigcup_{\bar{I} \in \mathsf{IP}} C_0^{\bar{I}} \cup \bigcup_{\bar{I} \in \mathsf{IP}} C_1^{\bar{I}}.$$

So we can take $x \in \mathcal{A}$ that avoids this set. This means σ is not winning strategy for the splitting* game with respect to $\mathcal{A}$. $\square$

Fact 7.3.10 ([IS88])**.** Assume CH. Then the splitting number $\mathfrak{s}$ is preserved under finite support iterations of Suslin ccc forcing.

Theorem 7.3.11. It is consistent relative to ZFC that $\mathfrak{s} < \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$.

Proof. By the previous fact, it is enough to show that there is a Suslin ccc forcing P that forces

$$(\exists \sigma \colon 2^{<\omega} \to 2)(\forall x \in 2^\omega \cap V)(\{k \in \omega : \sigma(x \restriction k) = 1\} \text{ is infinite and reaps } x).$$

Define such a forcing poset P as follows:

$$P = \{(n, s, H) : n \in \omega, s \colon 2^{<n} \to 2, H \subseteq 2^\omega \setminus \mathbb{0} \text{ finite}\}.$$

The order is as follows:

$$\begin{aligned}(n', s', H') \le (n, s, H) \iff & n \le n', s \subseteq s', H \subseteq H' \text{ and} \\ & (\forall x \in H)(\forall i \in [n, n'))(s'(x \restriction i) = 1 \to x(i) = 1).\end{aligned}$$

Define a P-name $\dot{\sigma}$ as follows:

$$\Vdash \dot{\sigma} = \bigcup \{s : (n, s, H) \in G\}.$$

It is clear that P is Suslin and σ-centered.

By the definition of the poset, it is also clear that

$$\Vdash (\forall x \in (2^\omega \setminus \mathbb{0}) \cap V)(\{k \in \omega : \dot{\sigma}(x \restriction k) = 1\} \subseteq^* x).$$

In the case $x \in \mathbb{0}$, $\omega \setminus x$ is almost the entire set ω, so it is clear that

$$\Vdash (\{k \in \omega : \dot{\sigma}(x \restriction k) = 1\} \subseteq^* \omega \setminus x).$$

In the rest, we show

$$\Vdash (\forall x \in 2^\omega \cap V)(\{k \in \omega : \dot{\sigma}(x \restriction k) = 1\} \text{ is infinite}).$$

Take a (V, P)-generic filter G. Fix $x \in 2^\omega \cap V$ and $l \in \omega$. Define the following subset E of P in V:

$$E = \{(n, s, H) \in P : (\exists i \in [l, n))(s(x \restriction i) = 1)\}.$$

We claim this is a dense set. To show this, fix $(n, s, H) \in P$. Take $n^* > \max\{n, l\}$ such that $y \restriction n^*$'s ($y \in H \cup \{x\}$) are pairwise different. If $x \notin \mathbb{0}$, take $n' > n^*$ so that $x(n') = 1$, otherwise put $n' = n^*$. Define $s' \colon 2^{<(n'+1)} \to 2$ that extends s and $s'(x \restriction n') = 1$. If the other extended parts are set to 0, we have $(n', s', H) \le (n, s, H)$ and $(n', s', H) \in E$.

Therefore, we can take $q = (n_q, s_q, H_q) \in G \cap E$ below p. Then there is $i \in [l, n_q)$ such that $s_q(x \restriction i) = 1$. So we have $\dot{\sigma}_G(x \restriction i) = 1$. $\square$

Remark 7.3.12. Using a forcing notion defined Section 4 of [GKMS21] before we force by the finite support iteration of P defined in the above theorem, we can also force that $\mathfrak{s}_\sigma < \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$. This is because the forcing in [GKMS21] adds σ-splitting family of size $\aleph_1$ that is not destroyed by finite support iteration of Suslin ccc forcing notions.

7.4 Reaping games

In this section, we consider reaping games and reaping* games, which are related to reaping families. The main result of this section is that $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{r}^{\mathrm{I}}_{\mathrm{game}} \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$. Of course, if $\mathfrak{r} = \mathfrak{r}_\sigma$ it would turn out that $\max\{\mathfrak{r}, \mathfrak{d}\} = \mathfrak{r}^{\mathrm{I}}_{\mathrm{game}}$. It is in fact an open question if it is consistent that $\mathfrak{r} \neq \mathfrak{r}_\sigma$. For more information the reader may want to look at [Bre98].

Fix a set $\mathcal{A} \subseteq [\omega]^\omega$. We call the following game the *reaping game* with respect to $\mathcal{A}$:

Player I	n_0		n_1		$\dots$	
Player II		i_0		i_1		$\dots$

Here, $n_0 < n_1 < n_2 < \dots < n_k < \dots$ are increasing numbers in ω, i_k $(k \in \omega)$ are elements in $\{0, 1\}$. Player II wins when there is $A \in \mathcal{A}$ such that

$$\{n_k : k \in \omega\} \cap A = \{n_k : k \in \omega \text{ and } i_k = 1\} \text{ and } A \text{ reaps } \{n_k : k \in \omega\}.$$

We call the following game the *reaping* game* with respect to $\mathcal{A}$:

Player I	i_0		i_1		$\dots$	
Player II		j_0		j_1		$\dots$

Here, $\langle i_k : k \in \omega \rangle$ and $\langle j_k : k \in \omega \rangle$ are sequences of elements in $\{0, 1\}$. Player II wins when Player II played 1 infinitely often and

$$\{k \in \omega : j_k = 1\} \in \mathcal{A} \text{ and reaps } \{k \in \omega : i_k = 1\}.$$

We define $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}}, \mathfrak{r}^{\mathrm{II}}_{\mathrm{game}}, \mathfrak{r}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{r}^{\mathrm{II}}_{\mathrm{game}^*}$ using reaping games and reaping* games in the same fashion as Definition 7.1.1.

Note that, when Player II wins in the reaping game, then the set $A \in \mathcal{A}$ that witnesses Player II wins satisfies the following condition.

(1) if $\langle i_k : k \in \omega \rangle$ is eventually zero, A is almost disjoint from $\langle n_k : k \in \omega \rangle$.

(2) if the digit 1 appears infinitely often in $\langle i_k : k \in \omega \rangle$, A is almost contained in $\langle n_k : k \in \omega \rangle$ and $A =^* \{n_k : i_k = 1\}$.

Theorem 7.4.1. $\mathfrak{r}^{\mathrm{II}}_{\mathrm{game}} = \mathfrak{c}$ holds.

Proof. Fix $\mathcal{A} \subseteq [\omega]^\omega$ such that Player II has a winning strategy for the reaping game with respect to $\mathcal{A}$. Fix such a strategy. We shall show that $\mathcal{A}$ is of size $\mathfrak{c}$. Consider the game tree $T \subseteq \omega^{<\omega}$ that the strategy determines.

First, assume the following.

- (Case 1) There is a $\sigma \in T$ of even length such that for every $m > \sigma(|\sigma|-2)$ there is $i_m < 2$ such that for every $\tau \in T$ extending σ and every $r \in [|\sigma|, |\tau|)$, $\tau(r) = m$ implies $\tau(r+1) = i_m$.

Fix a witness σ and $\langle i_m : m \geq \sigma(|\sigma|-2)\rangle$ of Case 1. If i_m's are eventually zero, clearly there is a play that Player II loses along the strategy, which is a contradiction.

So i_m's are not eventually zero. Take an infinite set $X \subseteq [\sigma(|\sigma|-2), \omega)$ such that $i_m = 1$ for every $m \in X$. Considering Player I plays an arbitrary subset of X, Player II must accordingly produce an $A \in \mathcal{A}$ that is almost equal to this set. But the cardinality of $[X]^\omega/\mathsf{fin}$ is $\mathfrak{c}$. So we have shown $|\mathcal{A}| = \mathfrak{c}$.

Next, we assume the negation of Case 1. Similar to the proof of Theorem 7.3.3, in this case we can construct a perfect subtree of T whose different paths yield different members of $\mathcal{A}$. $\square$

Theorem 7.4.2. $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}} \geq \mathfrak{r}, \mathfrak{d}$ holds.

Proof. That $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}} \geq \mathfrak{r}$ is easy. We show $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}} \geq \mathfrak{d}$. Fix a family $\mathcal{A}$ such that Player I has no winning strategy for the reaping game with respect to $\mathcal{A}$. For $A \in [\omega]^\omega$, let e_A be the increasing enumeration of A. Put

$$\mathcal{F} = \{e_B : B \text{ is almost equal to some } A \in \mathcal{A}\}.$$

Then we have $|\mathcal{F}| = |\mathcal{A}|$.

We shall show that $\mathcal{F}$ is a dominating family. So we fix an arbitrary increasing function $g \in \omega^\omega$. Let us consider the following strategy of Player I: First play $f(0)$. If Player II responds 0 then play $f(0)+1$, otherwise play $f(1)$. In general, if in the last time Player I played $f(l)+m$ and Player II responded 0, then play $f(l)+m+1$, otherwise, play $f(l+1)+m$.

By the assumption, this strategy is not a winning strategy, so there is a Player II's play $\bar{i} \in 2^\omega$ and $A \in \mathcal{A}$ such that A witnesses Player II wins with $\bar{i}$ against the strategy.

Let $\langle n_k : k \in \omega\rangle$ be the corresponding play of Player I. If $\bar{i}$ is eventually zero, then $\langle n_k : k \in \omega\rangle$ contains almost all integers in ω. Moreover, by the rule of the game, A is almost disjoint from this set. This cannot happen.

So the digit 1 appears infinitely often in $\bar{i}$. Then $A =^* \{n_k : i_k = 1\}$. Call the last set B. Then $e_B \in \mathcal{F}$ and e_B dominates f by the choice of the strategy.

Therefore, $\mathcal{F}$ is a dominating family. $\square$

Define a cardinal invariant $\mathfrak{r}_{\mathrm{simult}}$ as follows:

$$\begin{aligned}\mathfrak{r}_{\mathrm{simult}} = \min\{\mathcal{F} \subseteq ([\omega]^\omega)^\omega : (\forall \langle A_n \in [\omega]^\omega : n \in \omega\rangle)(\exists \langle B_n : n \in \omega\rangle \in \mathcal{F})\\ [(\exists n)(B_0 \subseteq \omega \setminus A_n) \text{ or } (\forall n)(B_n \subseteq A_n)]\}\end{aligned}$$

It can be easily seen that $\mathfrak{r}, \mathfrak{d} \leq \mathfrak{r}_{\mathrm{simult}}$.

Proposition 7.4.3. $\mathfrak{r}_{\mathrm{simult}} \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$.

Proof. Let $\mathcal{R}$ be a σ-reaping family of size $\mathfrak{r}_\sigma$ and $\mathcal{D}$ be a totally dominating family of ω^ω of size $\mathfrak{d}$. For $(C, h) \in \mathcal{R} \times \mathcal{D}$, we let

$$B_n^{C,h} = C \setminus h(n).$$

We now show $\{\langle B_n^{C,h} : n \in \omega\rangle : (C, h) \in \mathcal{R} \times \mathcal{D}\}$ is a witness of $\mathfrak{r}_{\mathrm{simult}}$. Fix a sequence $\langle A_n \in [\omega]^\omega : n \in \omega\rangle$. Since $\mathcal{R}$ is a σ-reaping family, we can take $C \in \mathcal{R}$ such that

$$(\forall n)(C \subseteq^* A_n \text{ or } C \subseteq^* \omega \setminus A_n).$$

We first consider the case $C \subseteq^* \omega \setminus A_n$ for some n. Take m such that $C \setminus m \subseteq \omega \setminus A_n$. Take $h \in \mathcal{D}$ such that $h(0) \geq m$. Then clearly, $\langle B_n^{C,h} : n \in \omega \rangle$ satisfies the condition of $\mathfrak{r}_{\mathrm{simult}}$.

We next consider the case $C \not\subseteq^* \omega \setminus A_n$ for every n. Then for every n, we have $C \subseteq^* A_n$. Let $f \in \omega^\omega$ be such that $C \setminus f(n) \subseteq A_n$. Take $h \in \mathcal{D}$ that totally dominates f. Then we also have $C \setminus h(n) \subseteq A_n$. Then $\langle B_n^{C,h} : n \in \omega \rangle$ satisfies the condition of $\mathfrak{r}_{\mathrm{simult}}$. □

Theorem 7.4.4. $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}} \leq \mathfrak{r}_{\mathrm{simult}}$ holds.

Proof. Fix a witness $\mathcal{F}$ of $\mathfrak{r}_{\mathrm{simult}}$.

Using a bijection between ω and $\omega^{<\omega}$, we think $\mathcal{F}$ is a subset of $([\omega]^\omega)^{(\omega^{<\omega})}$. That is, $\mathcal{F}$ satisfies the following condition:

$$\begin{aligned}&(\forall \langle A_t \in [\omega]^\omega : t \in \omega^{<\omega} \rangle)(\exists \langle B_t : t \in \omega^{<\omega} \rangle \in \mathcal{F}) \\ &\qquad [(\exists t)(B_\varnothing \subseteq \omega \setminus A_t) \text{ or } (\forall t)(B_t \subseteq A_t)]. \end{aligned} \tag{$*$}$$

Fix $\bar{B} = \langle B_t : t \in \omega^{<\omega} \rangle \in \mathcal{F}$. We define $\langle b_n^{\bar{B}} : n \in \omega \rangle$ by

$$\begin{aligned} b_0^{\bar{B}} &= \varnothing, \\ b_{n+1}^{\bar{B}} &= b_n^{\bar{B}} {}^\frown \langle \min B_{b_n^{\bar{B}}} \rangle. \end{aligned}$$

Put $\varphi(\bar{B}) = \operatorname{ran} \bigcup_n b_n^{\bar{B}}$.

Define $\mathcal{A}$ by

$$\mathcal{A} = \{\varphi(\bar{B}) : \bar{B} \in \mathcal{F}\} \cup \{X : \bar{B} \in \mathcal{F}, X \text{ and } B_0 \text{ are almost equal}\}.$$

Note that $|\mathcal{A}| \leq |\mathcal{F}|$.

We show that Player I has no winning strategy for the reaping game with respect to $\mathcal{A}$.

Let $\sigma \colon 2^{<\omega} \to \omega$ be an arbitrary strategy of Player I. Consider the tree $T \subseteq \omega^\omega$ defined as follows. $T \cap \omega^1 = \{\langle \sigma(\varnothing) \rangle, \langle \sigma(0) \rangle, , \langle \sigma(00) \rangle, \dots\}$. In general, the node whose label is $\sigma(s)$ has children whose labels are $\sigma(s {}^\frown \langle 1 \rangle {}^\frown \langle 0 \rangle^m)$ for $m \in \omega$.

We now put for each $t \in \omega^{<\omega}$

$$A_t = \begin{cases} \mathrm{succ}_{T_\sigma}(t) & (\text{if } t \in T) \\ \omega & (\text{otherwise}). \end{cases}$$

Then applying $(*)$, we can take $\langle B_t : t \in \omega^{<\omega} \rangle \in \mathcal{F}$ such that

$$(\exists t)(B_\varnothing \subseteq \omega \setminus A_t) \text{ or } (\forall t)(B_t \subseteq A_t).$$

Consider the former case: $(\exists t)(B_\varnothing \subseteq \omega \setminus A_t)$. Fix such a t. Then t must be in T. If $t = \varnothing$, consider the following play:

Player I	$\cdot$	$\cdot$	$\cdot$	$\dots$
Player II	0	0	0	$\dots$

The middle dots $(\cdot)$ in the first row mean the play along σ. Then $B_\varnothing$, which is in $\mathcal{A}$ is a witness that Player II wins. Indeed, if $\langle n_k : k \in \omega \rangle$ is the play of Player I, then $B_\varnothing \subseteq \omega \setminus \{n_k : k \in \omega\}$.

If $t \neq \varnothing$, take $s \in 2^{<\omega}$ such that the label of t is $\varphi(s)$. Consider the following play:

Player I	$\cdot$	$\cdot$	$\dots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\dots$
Player II	$s(0)$	$s(1)$	$\dots$	$s(\lvert s\rvert-1)$	1	0	0	$\dots$

Then a real almost equal to $B_\varnothing$, which is in $\mathcal{A}$ is a witness that Player II wins.

Consider the latter case $(\forall t)(B_t \subseteq A_t)$. Let $A = \varphi(\langle B_t : t \in \omega^{<\omega}\rangle)$. Enumerate A by $A = \{a_n : n \in \omega\}$ in ascending order. Take the unique m_0 such that $a_0 = \sigma(\langle 0\rangle^{m_0})$ and put $s_0 = \langle 0\rangle^{m_0}{}^\frown\langle 1\rangle$. By induction on k, take the unique m_k such that $a_k = \sigma(s_k{}^\frown\langle 0\rangle^{m_k})$ and put $s_{k+1} = s_k{}^\frown\langle 0\rangle^{m_k}{}^\frown\langle 1\rangle$. Put $\bar{i} = \bigcup_k s_k$, which is a play of Player II. Let $\langle n_k : k \in \omega\rangle$ be the corresponding play of Player I. That is $n_k = \sigma(\bar{i} \upharpoonright k)$. Then we have $A \in \mathcal{A}$ and $A = \{n_k : k \in \omega, i_k = 1\}$. So Player II wins.

Therefore, in either case, Player II wins. So σ is not a winning strategy. □

Because of the following theorem, the cardinal invariants regarding reaping* games are not worth considering.

Theorem 7.4.5. For every $\mathcal{A} \subseteq [\omega]^\omega$, Player I has a winning strategy for the reaping* game with respect to $\mathcal{A}$.

Proof. Consider the following strategy of Player I:

- Play 0 first.
- If Player II's previous play is 1, change the move from the previous Player I's play.
- Otherwise, play the same move as Player I's previous play.

It can be easily seen that this is a winning strategy. □

7.5 Anti-localizing games

In this section, we consider games related to the cardinal invariant $\mathrm{add}(\mathcal{N})$.

Let $\mathcal{C} = \{\varphi : \varphi \text{ is a function with domain } \omega \text{ that satisfies } \varphi(n) \in [\omega]^{n+1} \text{ for every } n \in \omega\}$. We call elements in $\mathcal{C}$ slaloms.

Fix a set $\mathcal{A} \subseteq \omega^\omega$. We call the following game the *anti-localizing game* with respect to $\mathcal{A}$:

Player I	a_0		a_1		$\dots$	
Player II		i_0		i_1		$\dots$

Here, $\langle a_k : k \in \omega\rangle$ is a sequence with $a_k \in [\omega]^{k+1}$ for every k and $\langle i_k : k \in \omega\rangle$ is a sequence of numbers in 2. Player II wins when Player II played 1 infinitely often and there is $x \in \mathcal{A}$ such that

$$\{k \in \omega : i_k = 1\} = \{k \in \omega : x(k) \notin a_k\}.$$

We call the following game the *anti-localizing* game* with respect to $\mathcal{A}$:

Player I	a_0		a_1		$\dots$	
Player II		n_0		n_1		$\dots$

Here, $\langle a_k : k \in \omega \rangle \in \mathcal{C}$ and $\langle n_k : k \in \omega \rangle$ is a sequence of numbers in ω. Player II wins when

$$\langle n_k : k \in \omega \rangle \in \mathcal{A} \text{ and } (\exists^\infty k)(n_k \notin a_k).$$

We define $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}}, \mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}}, \mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}^*}$ using anti-localizing games and anti-localizing*-games in the same fashion as Definition 7.1.1.

Theorem 7.5.1. $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}} = \mathrm{add}(\mathcal{N})$ holds.

Before proving this theorem, we recall the relationship between $\mathrm{add}(\mathcal{N})$ and slaloms.

Fact 7.5.2 ([Bar10, Theorem 4.11])**.** The following are equivalent.

(1) $\mathrm{add}(\mathcal{N}) \leq \kappa$.

(2) There is a family $\mathcal{A} \subseteq \omega^\omega$ of size $\leq \kappa$ such that $(\forall \varphi \in \mathcal{C})(\exists x \in \mathcal{A})(\exists^\infty n)(x(n) \notin \varphi(n))$ holds.

(3) There is a family $\mathcal{A} \subseteq \omega^\omega$ of size $\leq \kappa$ such that $(\forall f \in \mathcal{C}^\omega)(\exists x \in \mathcal{A})(\forall m)(\exists^\infty n)(x(n) \notin f(m)(n))$ holds.

Proof of Theorem 7.5.1. $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}} \geq \mathrm{add}(\mathcal{N})$ holds by Fact 7.5.2. We prove $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}} \leq \mathrm{add}(\mathcal{N})$. Take a witness $\mathcal{A}$ for (3) of Fact 7.5.2. Now we want to prove that Player I has no winning strategy for the anti-localizing game with respect to $\mathcal{A}$. Take a strategy $\sigma \colon 2^{<\omega} \to [\omega]^{<\omega}$ of Player I. Since $\mathcal{A}$ satisfies the condition in (3) of Fact 7.5.2, we can take $x \in \mathcal{A}$ such that $(\exists^\infty n)(x(k) \notin \sigma(\bar{i} \upharpoonright k)))$ for every $\bar{i} \in \mathbb{0}$.

We now put $\bar{i} \in 2^\omega$ by

$$i_k = \begin{cases} 1 & (\text{if } x(i) \notin \sigma(\bar{i} \upharpoonright k)) \\ 0 & (\text{otherwise}) \end{cases}$$

If $\bar{i} \in \mathbb{0}$, then $(\exists^\infty n)(x(k) \notin \sigma(\bar{i} \upharpoonright k))$ by the choice of x. But this fact and the choice of $\bar{i}$ imply $\bar{i} \notin \mathbb{0}$. It's a contradiction. So $\bar{i} \notin \mathbb{0}$. Therefore $\bar{i}$ is a play of Player II that wins against Player I's strategy σ. $\square$

Theorem 7.5.3. $\mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}} = \mathrm{cov}(\mathcal{M})$ holds.

Before proving this theorem, we recall the relationship between $\mathrm{cov}(\mathcal{M})$ and slaloms.

Fact 7.5.4 ([BJ95, Lemma 2.4.2])**.** The following are equivalent.

(1) $\mathrm{cov}(\mathcal{M}) \leq \kappa$.

(2) There is a family $\mathcal{A} \subseteq \omega^\omega$ of size $\leq \kappa$ such that $(\forall \varphi \in \mathcal{C})(\exists x \in \mathcal{A})(\forall n)(x(n) \notin \varphi(n))$ holds.

In addition, the following characterization is well-known.

Fact 7.5.5. The following are equivalent.

(1) $\kappa < \mathrm{cov}(\mathcal{M})$.

(2) Martin's axiom for countable posets with κ-many dense subsets.

Proof of Theorem 7.5.3. We first prove $\mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}} \leq \mathrm{cov}(\mathcal{M})$. Take a family $\mathcal{A} \subseteq \omega^\omega$ of size $\mathrm{cov}(\mathcal{M})$ that satisfies (2) of Fact 7.5.4. Then the strategy that says 1 always is a winning strategy for Player II.

We next prove $\mathrm{cov}(\mathcal{M}) \leq \mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}}$. Assuming $\kappa < \mathrm{cov}(\mathcal{M})$, we shall prove $\kappa < \mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}}$. Fix a family $\mathcal{A}$ of size κ. Take an arbitrary strategy τ of Player II. We show that τ is not a winning strategy.

We may assume that Player II plays the digit 1 infinitely often along τ, otherwise, τ is clearly not a winning strategy.

Set $P = \bigcup_n \prod_{i<n} [\omega]^{i+1}$. For each $x \in \mathcal{A}$, we define a set D_x as follows:

$$D_x = \{p \in P : (\exists k \in \mathrm{dom}(p))(x(k) \in p(k) \text{ and } \tau(p \restriction (k+1)) = 1\}.$$

Then each D_x is a dense subset of P, using the above assumption.

Therefore, by Fact 7.5.5, we can take a filter $G \subseteq P$ that intersects with all D_x's. Put $g = \bigcup G$. Then if Player I plays g, then Player I wins against Player II, who obeys the strategy τ. ☐

Theorem 7.5.6. $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}^*} = \mathrm{add}(\mathcal{N})$ holds.

Proof. Using terminology in [Bla10, Section 10], $\mathrm{add}(\mathcal{N})^{\mathrm{I}}_{\mathrm{game}^*}$ is equal to the global, adaptive, prediction specified by the predefined function version of evasion number. Moreover, in the article, it was shown that this invariant is equal to $\mathrm{add}(\mathcal{N})$. ☐

Theorem 7.5.7. $\mathrm{add}(\mathcal{N})^{\mathrm{II}}_{\mathrm{game}^*} = \mathfrak{c}$ holds.

Proof. Fix $\mathcal{A} \subseteq \omega^\omega$ such that Player II has a winning strategy τ for the anti-localizing* game with respect to $\mathcal{A}$. We shall show that $\mathcal{A}$ is of size $\mathfrak{c}$. Consider the game tree $T \subseteq \prod_{n<\omega} X(n)$ that τ determines, where $X(2n) = [\omega]^{<n+1}$ and $X(2n+1) = \omega$.

First, assume the following.

- (Case 1) There is a $\sigma \in T$ such that for every odd $k \geq |\sigma|$, there is an $n_k < \omega$ such that for every $\tau \in T$ extending σ with $|\tau| > k$, we have $\tau(k) = m_k$.

Fix the witness $\sigma, \langle n_k : k \geq |\sigma| \rangle$ for Case 1.

Consider the next play.

Player I	$\sigma(0)$		$\dots$	$\sigma(\lvert\sigma\rvert-2)$		$\{n_{\lvert\sigma\rvert}\}$		$\{n_{\lvert\sigma\rvert+2}\}$		$\dots$
Player II		$\sigma(1)$	$\dots$		$\sigma(\lvert\sigma\rvert-1)$		$n_{\lvert\sigma\rvert}$		$n_{\lvert\sigma\rvert+2}$	

Then the sequence defined by the play of Player II does not avoid the slalom defined by the play of Player I. So Player II loses. This is a contradiction.

So Case 1 is false. Thus we have

- (Case 2) For every $\sigma \in T$, there is an odd number $k \geq |\sigma|$ such that for every $n < \omega$, there is $\tau \in T$ extending σ with $|\tau| > k$ such that $\tau(k) \neq n$.

Note that there are $\tau_0, \tau_1 \supseteq \sigma$ with $|\tau_0|, |\tau_1| > k$ such that $\tau_0(k) \neq \tau_1(k)$ in Case 2.

Now we can construct a subtree of T in the following manner. First we put $\sigma_\varnothing = \varnothing$. Suppose we have $\langle \sigma_s : s \in 2^{\leq l} \rangle$. Then for each $s \in 2^l$, we can take $\sigma_{s^\frown 0}, \sigma_{s^\frown 1} \supseteq \sigma_s$ and $k_s \geq |\sigma_s|$ such that $\sigma_{s^\frown 0}(k_s) \neq \sigma_{s^\frown 1}(k_s)$.

Now for each $f \in 2^\omega$, we put σ_f by $\sigma_f = \bigcup_{n\in\omega} \sigma_{f\restriction n}$.

For each $f \in 2^\omega$, we have $\sigma_f \in [T]$. So Player II wins at the play σ_f. So by the definition of the game, we can take $x_f \in \mathcal{A}$ such that $x_f(k) = \sigma_f(2k+1)$. It should be clear that if f and g are distinct elements of 2^ω, then we have $x_f \neq x_g$. Therefore we have $|\mathcal{A}| = \mathfrak{c}$. $\square$

7.6 Open problems

Question 7.6.1. Does it hold that $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{E})$, where $\mathcal{E}$ is the σ-ideal generated by closed null sets?

Question 7.6.2. What is the value of $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ in the model obtained by finite support iteration of the random forcing over a model of CH? (Note that in this model $\mathrm{non}(\mathcal{E})$ is small and $\mathrm{non}(\mathcal{M})$, $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$ are large.)

Question 7.6.3. Is there a lower bound of $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ other than $\mathfrak{s}_\sigma$? In particular, is $\mathrm{add}(\mathcal{N})$ a lower bound of $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$?

Question 7.6.4. Does ZFC prove that $\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}}$ is equal to $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$?

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