Goldstern's principle about unions of null sets

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Goldstern's theorem

(full domination order) For $x, x' \in \omega^\omega$, define a relation $x \leq x'$ by $(\forall n \in \omega)(x(n) \leq x'(n))$.

In 1993, Martin Goldstern proved the following theorem.

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Let (Y, μ) be a Polish probability space. Let $A \subseteq \omega^\omega \times Y$ be a $\mathbf{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has measure 0. Also, assume $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also measure 0.

He used the Shoenfield absoluteness theorem and the random forcing to show this theorem.

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The principle GP(Γ)

Definition

Let Γ be a pointclass. Then GP(Γ) means the following statement: Let (Y, μ) be a Polish probability space and $A \subseteq \omega^\omega \times Y$ be in Γ. Assume that for each $x \in \omega^\omega$, A_x has measure 0. Also suppose that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also measure 0.

Goldstern's theorem says that GP($\boldsymbol{\Sigma}^1_1$) holds.
Note that if Γ is a sufficiently high pointclass (that is if $\boldsymbol{\Delta}^1_1 \subseteq \Gamma$), then we can assume that (Y, μ) is the Cantor space with the standard measure.

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The principle $\mathrm{GP}^*(\Gamma)$

(almost domination order) For $x, x' \in \omega^\omega$, define a relation $x \leq^* x'$ by $(\exists m \in \omega)(\forall n \geq m)(x(n) \leq x'(n))$.

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Clearly $\mathrm{GP}(\Gamma) \Rightarrow \mathrm{GP}^*(\Gamma)$.

Lemma

If a pointclass Γ is closed under recursive substitution and projection along ω, then $\mathrm{GP}^*(\Gamma) \Rightarrow \mathrm{GP}(\Gamma)$.

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Main Result

The symbol “all” denotes the class of all subsets of Polish spaces.

Theorem

GP(all) is independent from ZFC.

Consistency of $\neg\,$GP(all)

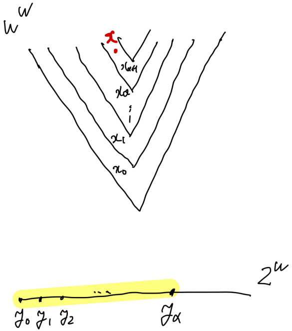
Theorem

Assume CH. Then $\neg\,$GP(all) holds.

Proof. Let $\langle x_\alpha : \alpha < \omega_1\rangle$ be a cofinal increasing sequence in $(\omega^\omega, <^*)$. And let $\langle y_\alpha : \alpha < \omega_1\rangle$ be an enumeration of 2^ω. Then the set A defined by the following equation witnesses $\neg\,$GP(all):

$$A_x = \{y_\beta : \beta < \alpha_x\},$$

where $\alpha_x = \min\{\alpha : x <^* x_\alpha\}$. □



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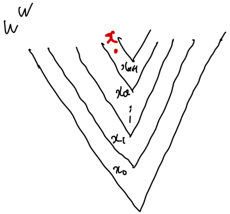

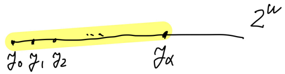

Consistency of $\neg\,\mathrm{GP}(\mathrm{all})$

Refining the last proof, we get the following theorem.

Theorem

Assume that at least one of the following three conditions holds:

$\mathrm{add}(\mathcal{N}) = \mathfrak{b}$, $\mathrm{non}(\mathcal{N}) = \mathfrak{b}$ or $\mathrm{non}(\mathcal{N}) = \mathfrak{d}$.

Then $\neg\,\mathrm{GP}(\mathrm{all})$ holds.

$\mathrm{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$
$\mathrm{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$
$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F)\ f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F)\ g <^* f\}$

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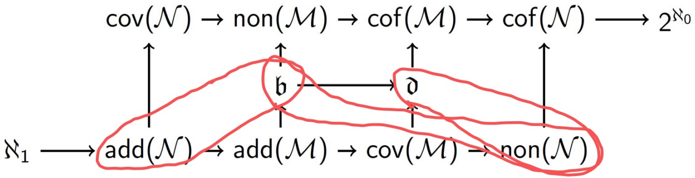

Consistency of GP(all)

Theorem

If ZFC is consistent then so is ZFC + GP(all).

In fact, "The Laver model" satisfies GP(all).

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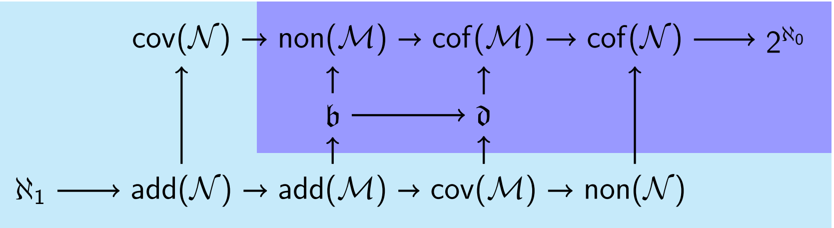

equal to $\aleph_1$ in the Laver model

Null tower

Definition (null tower)

We call a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ a **null tower** if it is an increasing sequence of measure 0 sets such that its union does not have measure 0.

Lemma

Assume that $\mathfrak{b} = \mathfrak{d}$ and let both of these be κ. Then the following are equivalent.

1. There is a null tower of length κ.
2. $\neg$ GP(all).

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Laver forcing

Definition (Laver forcing)

$$\mathbb{L} = \{p : p \text{ is a perfect subtree of } \omega^{<\omega} \text{ and all nodes in } p \text{ above the stem have infinitely many successors}\}$$

Elements in $\mathbb{L}$ are ordered by the inclusion.

Property of Laver forcing

- $\mathbb{L}$ is proper.
- $\mathbb{L}$ adds a dominating real.
- $\mathbb{L}$ (and csi of $\mathbb{L}$) preserve Lebesgue outer measure.

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Reflection Lemma

Lemma

Assume CH. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration of proper forcing notions such that

$$\Vdash_\alpha |\dot{Q}_\alpha| \leq \mathfrak{c} \quad \text{(for all } \alpha < \omega_2\text{)}.$$

Let $\langle \dot{X}_\alpha : \alpha < \omega_2 \rangle$ be a sequence of P_{ω_2}-names such that

$$\Vdash_{\omega_2} (\forall \alpha < \omega_2)(\dot{X}_\alpha \text{ has measure } 0).$$

Then the set

$$\begin{aligned} S = \{\alpha < \omega_2 : \operatorname{cf}(\alpha) = \omega_1 \text{ and} \\ \Vdash_{\omega_2} (\langle \dot{X}_\beta \cap V[\dot{G}_\alpha] : \beta < \alpha \rangle \in V[\dot{G}_\alpha] \text{ and} \\ (\forall \beta < \alpha)(\dot{X}_\beta \cap V[\dot{G}_\alpha] \text{ has measure } 0)^{V[\dot{G}_\alpha]})\} \end{aligned}$$

is stationary in ω_2.

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Then

$$\Vdash_{\omega_2} \mathsf{GP}(\mathsf{all}).$$

In particular, if ZFC is consistent then so is $\mathrm{ZFC} + \mathsf{GP}(\mathsf{all})$.

Proof of Main theorem (1/2)

- By the fact that $\Vdash_{\omega_2} \mathfrak{b} = \mathfrak{d} = \omega_2$, it is sufficient to show that

$$\Vdash_{\omega_2} \text{"There is no null tower of length } \omega_2\text{"}.$$

- Let G be a (V, P_{ω_2})-generic filter. In $V[G]$, consider an increasing sequence $\langle A_\alpha : \alpha < \omega_2 \rangle$ of measure 0 sets.
- By the lemma, we can find a stationary set $S \subseteq \omega_2$ such that for all $\alpha \in S$, $\mathrm{cf}(\alpha) = \omega_1$ and $(\langle A_\beta \cap V[G_\alpha] : \beta < \alpha \rangle \in V[G_\alpha]$ and $(\forall \beta < \alpha)((A_\beta \cap V[G_\alpha]$ has measure $0)^{V[G_\alpha]})$.
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Proof of Main theorem (2/2)

- Fix $\alpha \in S$. We now prove that B_α is also a measure 0 set in $V[G_\alpha]$. Let α' be the successor of α in S. Then B_α is a measure 0 set in $V[G_{\alpha'}]$. Since the quotient forcing $P_{\alpha'}/G_\alpha$ is a countable suppport iteration of the Laver forcing, this forcing preserves positive outer measure. So B_α is also a measure 0 set in $V[G_\alpha]$.
- For each $\alpha \in S$, take an Borel code $c_\alpha \in \omega^\omega$ of a measure 0 set such that $B_\alpha \subseteq \hat{c}_\alpha$ in $V[G_\alpha]$. Since $\mathrm{cf}(\alpha) = \omega_1$, each c_α appears a prior stage. Then by Fodor's lemma, we can take a stationary set $S' \subseteq \omega_2$ that is contained by S and $\beta < \omega_2$ such that $(\forall \alpha \in S')(c_\alpha \in V[G_\beta])$.
- But the number of reals in $V[G_\beta]$ is $\aleph_1$, so we can take $S'' \subseteq S'$ unbounded in ω_2 and c such that $(\forall \alpha \in S'')(c_\alpha = c)$. Then we have $\bigcup_{\alpha<\omega_2} A_\alpha \subseteq \hat{c}$ in $V[G]$. So $\bigcup_{\alpha<\omega_2} A_\alpha$ has measure 0. □

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Related results (1)

Theorem

Assume $\mathrm{ZF} + \mathrm{AD}$. Then GP(all) holds.

Corollary (of the local version of the theorem)

1. If ZFC + "there is a measurable cardinal" is consistent, then so is $\mathrm{ZFC} + \mathrm{GP}(\boldsymbol{\Sigma}^1_2) + \neg\,\mathrm{GP}(\boldsymbol{\Delta}^1_3)$.
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In Solovay models, GP(all) holds.

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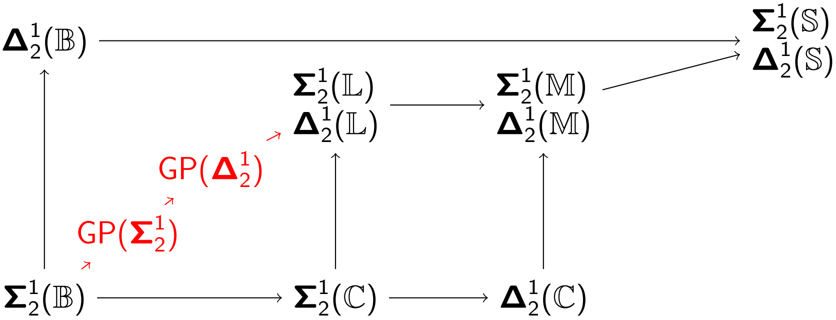

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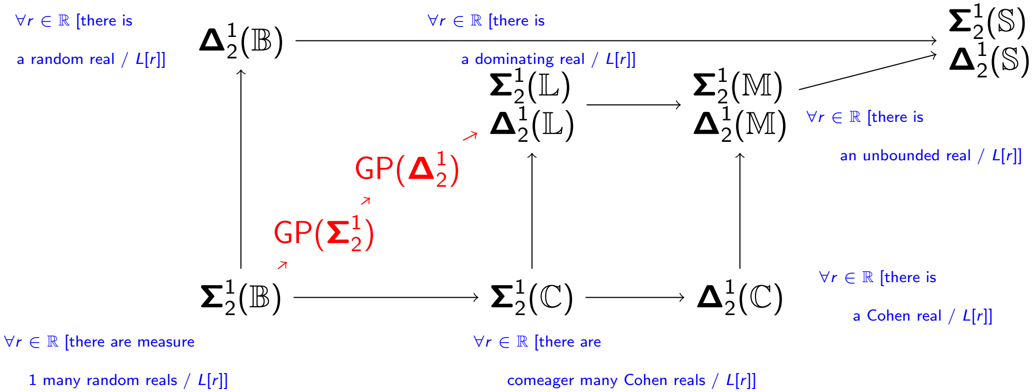

Open questions

1. Is $\mathrm{ZFC} + (\mathfrak{c} \geq \aleph_3) + \mathrm{GP}(\text{all})$ consistent?
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3. Does $V = L$ imply $\neg\,\mathrm{GP}(\Pi^1_1)$?
4. (Assuming an inaccessible cardinal) is there a model of ZF satisfying that every set of reals is Lebesgue measurable and $\neg\,\mathrm{GP}(\text{all})$ holds?
5. For some $n \geq 2$ (or for every $n \geq 2$), can we separate $\mathrm{GP}(\boldsymbol{\Sigma}^1_{n+1})$ and $\mathrm{GP}(\boldsymbol{\Sigma}^1_n)$ without using large cardinals?
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We think it holds in the model obtained by adding $\aleph_3$ random reals over Laver model.

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We considered the possibility that $\mathrm{GP}(\mathrm{all})$ implies $\mathfrak{b} = \mathfrak{d}$, but it did not work. If we consider this consistency to be true, then this is a more difficult problem than the first item.

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Perheps ZFC proves $\mathrm{GP}(\boldsymbol{\Pi}^1_1)$. Actually, for a set A satisfying the assumption of $\mathrm{GP}(\boldsymbol{\Pi}^1_1)$, isn't $\bigcup_{x \in \omega^\omega} A_x$ provably-$\boldsymbol{\Delta}^1_2$?

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Can we use the model or the idea used in Shelah's consistency proof for $\mathrm{LM} + \neg\mathrm{BP}$?

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For $n = 2$, we should consider the model obtained by forcing MA over L. The idea of the Raisonnier filter could be used.

Or Harrington's model of $\mathsf{MA} + (\mathfrak{c} = \aleph_2) +$ "there is $\boldsymbol{\Delta}^1_3$ wellorder of $\mathbb{R}$" could be used.

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We should consider the Hechler model first.

References and acknowledgments

[Gol93] Martin Goldstern. "An Application of Shoenfield's Absoluteness Theorem to the Theory of Uniform Distribution.". In: *Monatshefte für Mathematik* 116.3-4 (1993), pp. 237–244.

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