Game-theoretic variants of cardinal invariants

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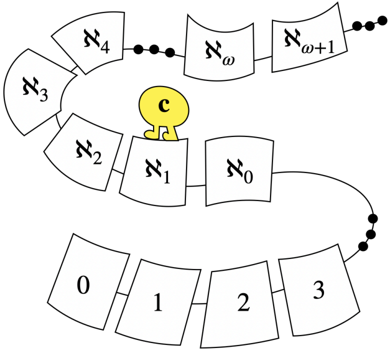
Set theory

Set theory is the field where we consider various aspects of infinite sets, in particular their cardinalities.

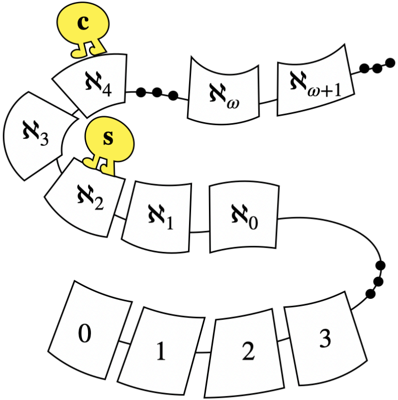
$\aleph_0$ denotes the cardinality of a countable set and $\aleph_1$ denotes the successor cardinal of $\aleph_0$, and so on.
$\mathbf{c}$ means the cardinality of the continuum.

$\aleph_0 < \mathbf{c}$ is a theorem of ZFC (Cantor).
Whether $\mathbf{c}$ is $\aleph_1$ or not cannot be determined by ZFC (Gödel, Cohen).



Cardinal invariants

Cardinal invariants on the continuum are cardinals defined from the structure of the reals. They lie typically between $\aleph_1$ and $\mathbf{c}$.
Many of them are neither shown to equal to $\aleph_1$ nor shown to equal to $\mathbf{c}$ in ZFC.



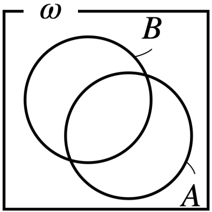
Infinite games

Games between two players of length ω are very important in set theory.

In particular, the axiom of determinacy is an important axiom about infinite games, but it is incompatible to the axiom of choice. In this presentation we will not consider the axiom of determinacy, but assume the axiom of choice as usual.

This study connects the two fields of cardinal invariants and game theory by examining what can be obtained from game-theoretic modifications of cardinal invariants.

The definition of the splitting number



For infinite subsets A, B of ω, we say A **splits** B if

$$|B \cap A| = |B \setminus A| = \aleph_0.$$

For $\mathcal{S} \subseteq [\omega]^\omega$, we say

- $\mathcal{S}$ is a **splitting family**

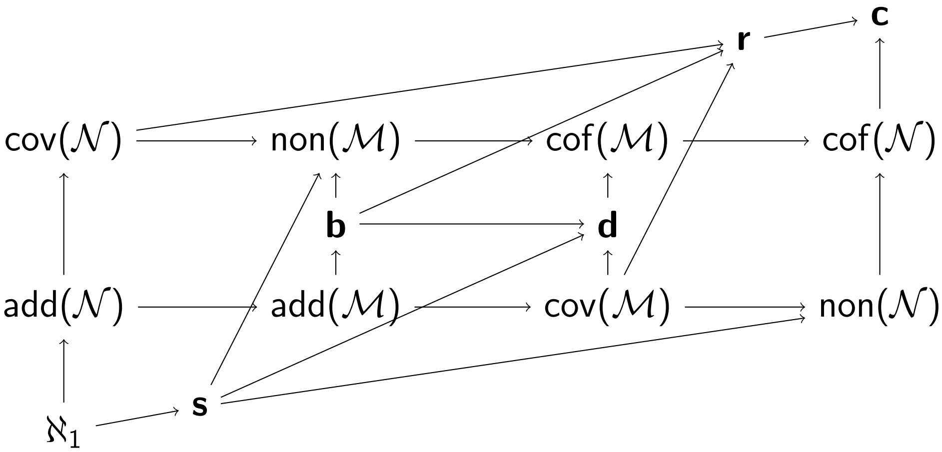
  $$:\iff (\forall B \in [\omega]^\omega)(\exists A \in \mathcal{S})(A \text{ splits } B).$$

The cardinal $\mathbf{s}$ defined below is called the splitting number:

- $\mathbf{s} := \min\{|\mathcal{S}| : \mathcal{S} \text{ is a splitting family}\}.$

$\mathfrak{s}$ and cardinal invariants

$\mathfrak{s}$ is a typical example of cardinal invariants of the continuum.



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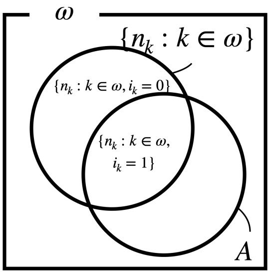
Splitting games

Fix $\mathcal{A} \subseteq [\omega]^\omega$. Call the following game the **splitting game** with respect to $\mathcal{A}$:

Player I	n_0	$<$	n_1	$<$	$\dots$		
Player II		$i_0 \in 2$		$i_1 \in 2$		$\dots$	



Player II wins $\Leftrightarrow$ Player II played both 0 and 1 infinitely and there is $A \in \mathcal{A}$ such that

$$\{n_k : k \in \omega\} \cap A = \{n_k : k \in \omega \text{ and } i_k = 1\}.$$

Cardinal invariants on splitting games

Definition

$$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} = \mathsf{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting game with respect to } \mathcal{A}, \text{ Player I does not have a winning strategy}\}$$

$$\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} = \mathsf{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting game with respect to } \mathcal{A}, \text{ Player II has a winning strategy}\}$$

Theorems about splitting games

We can easily see the following.

Proposition (Chapital–G.–Hayashi)

$\mathbf{s} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{II}}_{\mathrm{game}} \leq \mathbf{c}$.

The following needs some discussion.

Theorem (Chapital–G.–Hayashi)

$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} = \mathbf{s}_\sigma$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} = \mathbf{c}$.

(We will see the definition of $\mathbf{s}_\sigma$ in the next page.)

Definition of σ-splitting number

For $A \in [\omega]^\omega$ and $f : \omega \to [\omega]^\omega$, we say A σ-splits f if

$$\text{For every } n,\ A \text{ splits } f(n).$$

For $\mathcal{S} \subseteq [\omega]^\omega$, we say $\mathcal{S}$ is a **σ-splitting family**

$$:\iff (\forall f : \omega \to [\omega]^\omega)(\exists A \in \mathcal{S})(A\ \sigma\text{-splits } f).$$

We call the following cardinal the σ-splitting number:

- $\mathbf{s}_\sigma := \min\{|\mathcal{S}| : \mathcal{S} \text{ is a } \sigma\text{-splitting family}\}$.

It can be easily seen that $\mathbf{s} \leq \mathbf{s}_\sigma$.

Note It is a longstanding open question whether ZFC proves $\mathbf{s} = \mathbf{s}_\sigma$!

The proof of $\mathfrak{s}_\sigma \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}}$ (1/2)

Theorem $\mathfrak{s}_\sigma \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}}$.

Proof. Fix a family $\mathcal{A} \subseteq [\omega]^\omega$ such that Player I has no winning strategy for the splitting game with respect to $\mathcal{A}$. We want to show that $\mathcal{A}$ is a σ-splitting family.

Take $f : \omega \to [\omega]^\omega$. We shall find an $A \in \mathcal{A}$ such that A splits $f(n)$ for every $n \in \omega$. We may assume that each element of $\mathrm{ran}(f)$ appears infinitely in the sequence f. Let a^n_m denote the m-th element of $f(n)$.

The proof of $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$ (2/2)

Theorem $\mathbf{s}_\sigma \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}}$.

Consider the following strategy σ of Player I. First σ plays a^0_0. From then on, σ will says the elements of $f(0)$ in turn until Player II says both 0 and 1. After that, σ says a^1_k next. Here k is the smallest number such that a^1_k exceeds the number that σ has said so far. Continue this process.

Player I	a^0_0		a^0_1		a^0_2		a^1_k		$\ldots$
Player II		0		0		1			$\ldots$

Since this σ is not a winning strategy, there is $A \in \mathcal{A}$ and the play $\vec{i} \in 2^\omega$ such that Player II wins with $\vec{i}$. This implies A splits all elements in $\mathrm{ran}(f)$ by the definition of σ. □

splitting* game

Fix $\mathcal{A} \subseteq [\omega]^\omega$. Call the following game **splitting* game** with respect to $\mathcal{A}$.

Player I	i_0		i_1		$\dots$	
Player II		j_0		j_1		$\dots$

Both $i_0, i_1, \dots, i_k, \dots$ and $j_0, j_1, \dots, j_k, \dots$ are binary sequences. Player II wins if either Player I said 1 finitely or

$$\{k \in \omega : j_k = 1\} \text{ is in } \mathcal{A} \text{ and splits } \{k \in \omega : i_k = 1\}.$$

Cardinal invariants on splitting* games

Definition

$$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting* game with respect to } \mathcal{A}, \text{ Player I does not have a winning strategy}\}$$

$$\mathbf{s}^{\mathrm{II}}_{\mathrm{game}^*} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega), \text{ In the splitting* game with respect to } \mathcal{A}, \text{ Player II has a winning strategy}\}$$

An observation of splitting* game

The splitting* game with respect to $\mathcal{A}$ is a harder game for Player II than the splitting game with respect to $\mathcal{A}$.

Therefore, $\mathbf{s}^{\mathrm{I}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}} \leq \mathbf{s}^{\mathrm{II}}_{\mathrm{game}^*}$ hold.

Thus, we have $\mathbf{s}_{\sigma} \leq \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathbf{s}^{\mathrm{II}}_{\mathrm{game}^*} = \mathbf{c}$.

Theorems about splitting* game

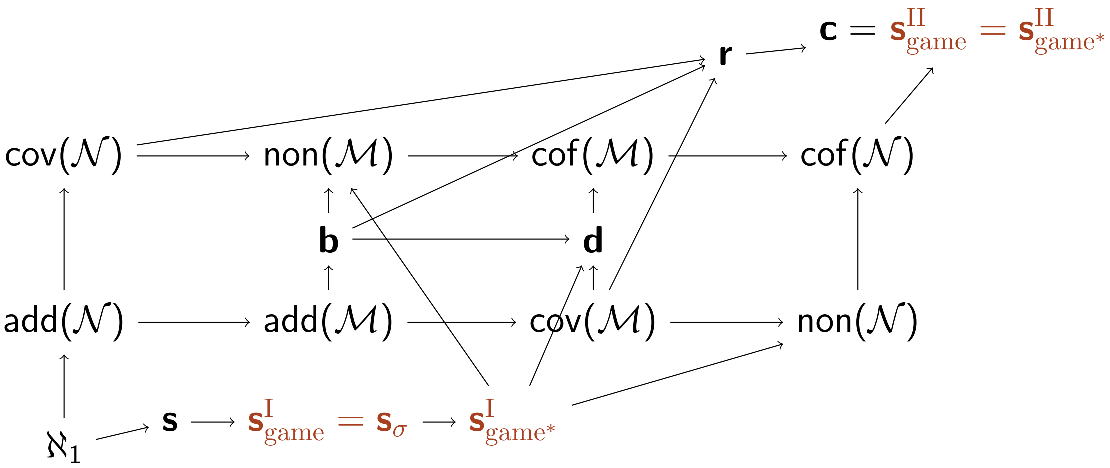
Theorem (Chapital–G.–Hayashi)

The proposition $\mathbf{s} < \mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*}$ is relatively consistent from ZFC.

Theorem (Chapital–G.–Hayashi)

$\mathbf{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathrm{non}(\mathcal{M}), \mathbf{d}, \mathrm{non}(\mathcal{N})$.

The diagram with invariants regarding splitting games



Other games

game	$\mathfrak{r}^{\mathrm{I}}_{\mathrm{game}}$	$\mathfrak{r}^{\mathrm{II}}_{\mathrm{game}}$
splitting	$\mathbf{s}_\sigma$	$\mathbf{c}$
splitting*	$\mathbf{s}_\sigma \leq ? \leq \min\{\mathrm{non}(\mathcal{M}), \mathbf{d}, \mathrm{non}(\mathcal{N})\}$	$\mathbf{c}$
reaping	$\max\{\mathbf{r}, \mathbf{d}\} \leq ? \leq \max\{\mathbf{r}_\sigma, \mathbf{d}\}$	$\mathbf{c}$
reaping*	∞	∞
bounding	$\mathbf{b}$	$\mathbf{d}$
bounding*	$\mathbf{b}$	$\mathbf{c}$
dominating	$\mathbf{d}$	$\mathbf{d}$
dominating*	$\mathbf{d}$	$\mathbf{c}$
anti-localizing	$\mathrm{add}(\mathcal{N})$	$\mathrm{cov}(\mathcal{M})$
anti-localizing*	$\mathrm{add}(\mathcal{N})$	$\mathbf{c}$

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Tallness games

Let $\mathcal{I}$ be an ideal on ω. We call the following game the **tallness game** with respect to $\mathcal{I}$:

Player I	n_0	$<$	n_1	$<$	$\dots$	
Player II		$i_0 \in 2$		$i_1 \in 2$		$\dots$

Player II wins when

$$\{n_k : k \in \omega, i_k = 1\} \in \mathcal{I} \cap [\omega]^\omega.$$

Theorems regarding tallness games

Theorem (Chapital–G.–Hayashi)

The following are equivalent:

1. Player II has a winning strategy for the tallness game with respect to $\mathcal{I}$.
2. $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{I}$.

Theorem (Chapital–G.–Hayashi)

The following are equivalent:

1. Player I has a winning strategy for the tallness game with respect to $\mathcal{I}$.
2. $\mathrm{non}^*(\mathcal{I}) = \aleph_0$.

Theorems regarding tallness games

The following dichotomy follows from the 2 theorems on the previous page:

For a Borel ideal $\mathcal{I}$, we have either $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{I}$ or $\mathrm{non}^*(\mathcal{I}) = \aleph_0$.

This is the fact already proved by Hrušák–Meza–Minami [HMM10]. In fact, the tallness game is equivalent to the game they use.

+tallness games

We modify tallness games and consider **+tallness games**.

Player I	n_0	$<$	n_1	$<$	$\dots$		
Player II		$i_0 \in 2$		$i_1 \in 2$		$\dots$	

Player II wins if we have either:

- $\{n_k : k \in \omega\} \in \mathcal{I}$ or
- $\{n_k : k \in \omega\} \in \mathcal{I}^+$ and $\{n_k : k \in \omega, i_k = 1\} \in \mathcal{I} \cap [\omega]^\omega$.

Using this game, we obtained another proof of The Category Dichotomy.

The Category Dichotomy (Hrušák)

For a Borel ideal $\mathcal{I}$, we have either $\mathcal{I} \leq_{\mathrm{K}}$ nwd or $\mathcal{I} \restriction X \geq_{\mathrm{K}} \mathcal{ED}$ for some $\mathcal{I}$-positive set X.

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